

# Statistical anisotropy and cosmological quantum relaxation

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We show that cosmological quantum relaxation predicts an anisotropic primordial power spectrum with a specific dependence on wavenumber  $k$ . We explore some of the consequences for precision measurements of the cosmic microwave background (CMB). Quantum relaxation is a feature of the de Broglie-Bohm pilot-wave formulation of quantum theory, which allows the existence of more general physical states that violate the Born probability rule. Recent work has shown that relaxation to the Born rule is suppressed for long-wavelength field modes on expanding space, resulting in a large-scale power deficit with a characteristic inverse-tangent dependence on  $k$ . Because the quantum relaxation dynamics is independent of the direction of the wave vector for the relaxing field mode, in the limit of weak anisotropy we are able to derive an expression for the anisotropic power spectrum that is determined by the power deficit function. As a result, the off-diagonal terms in the CMB covariance matrix are also determined by the power deficit. We show that the lowest-order  $l - (l + 1)$  inter-multipole correlations have a characteristic scaling with multipole moment  $l$ . Our derived spectrum also predicts a residual statistical anisotropy at small scales, with an approximate consistency relation between the scaling of the  $l - (l + 1)$  correlations and the scaling of the angular power spectrum at high  $l$ . We also predict a relationship between the  $l - (l + 1)$  correlations at large and small scales. Cosmological quantum relaxation appears to provide a single physical mechanism that predicts both a large-scale power deficit and a range of statistical anisotropies, together with potentially testable relationships between them.

# 1 Introduction

Tentative evidence has accumulated for the existence of primordial anomalies as deduced from observations of the cosmic microwave background (CMB), in particular a large-scale power deficit together with a range of violations of statistical isotropy (at both large and small scales). An exhaustive analysis and survey has recently appeared in the *Planck* 2015 data release [1, 2, 3]. As emphasised by the Planck team, in the absence of an established theoretical framework that predicts such effects the statistical significance of the anomalies remains difficult to assess. Are they mere fluctuations or do they reflect new physics? The absence of established predictive models also makes it difficult to judge what to look for in such a large and multi-faceted data set. However, if a single mechanism could be found that explains several of the anomalies at once as well as predicting relationships between them, one might obtain a more decisive statistical significance (for or against the model).

Some theoretical models address the large-scale power deficit. This anomaly might be explained by corrections to the quantum inflationary vacuum state induced by a radiation-dominated pre-inflationary phase [4, 5], by a suitable period of fast rolling for the inflaton field [6, 7], or by a form of multifield inflation [8]. Other models address the statistical anisotropy. This multi-faceted anomaly, or aspects of it, might be explained in a number of ways, such as a marginally-open universe with a finite curvature scale (which can induce a dipolar modulation of the temperature anisotropy) [9] or the pre-inflationary decay of a false vacuum with a smaller number of large dimensions than the current vacuum [10]. Statistical anisotropy can also arise from a coupling between the inflaton and a vector field [11, 12, 13]. Power asymmetry can be generated by a multifield inflationary (curvaton) model [14] or by a primordial domain wall [15]. Finally, a few models address both kinds of anomaly. Both a power deficit and a statistical anisotropy can be generated by a strongly-anisotropic pre-inflationary phase described by a Kasner geometry [16] or by a direction-dependent inflaton field with an induced anisotropic Hubble parameter [17]. It remains to be seen if any of these models can provide a fully satisfactory account of the observed anomalies and if the detailed predictions they make stand up to close comparison with the data.

In this paper we are concerned with a physical mechanism – cosmological quantum relaxation [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28] – which predicts both a power deficit and statistical anisotropy at large scales, as well as a residual statistical anisotropy at small scales, and which furthermore predicts testable relationships between the two kinds of anomaly.

The de Broglie-Bohm pilot-wave formulation of quantum theory [29, 30, 31, 32, 33] provides a natural generalisation of the usual quantum formalism, with probabilities potentially different from those predicted by the Born rule [18, 19, 20, 21, 22, 34]. On this view, the Born rule is not a law but only describes a particular state of statistical equilibrium. Conventional quantum physics is then merely an effective theory of ‘quantum equilibrium’. A wider physics of quantum nonequilibrium – with violations of the Born rule and associated new phenomena

– is possible in principle [18, 19, 20, 21, 22, 35, 23, 24, 36, 25, 26, 37, 34]. This wider physics may have existed in the early universe [18, 19, 20, 21, 22], in which case it could leave discernible traces in the CMB [23, 26, 27, 28] and possibly even survive until today for some relic cosmological particles [22, 23, 24, 38].

In pilot-wave theory, one may consider an expanding universe that begins in a state of quantum nonequilibrium – that is, with non-Born rule probabilities at the earliest times [20, 21, 22, 23, 24, 26]. If one begins with such a state, for example for a representative scalar field on expanding space, the dynamics of pilot-wave field theory allows us to evolve the state forward in time. Extensive numerical simulations have shown that short-wavelength field modes evolve rapidly to the Born rule [39, 40, 41, 42, 43]. We would therefore expect local laboratory physics to obey the Born rule. This expectation is consistent with what is observed in (for example) atomic physics and high-energy scattering experiments. However, a combination of theoretical analysis and further simulations has shown that long-wavelength (super-Hubble) field modes evolve more slowly and need not reach equilibrium [23, 24, 26, 27, 28].

If one takes pilot-wave theory seriously on its own terms, one may reasonably expect the existence of departures from the Born rule at the longest wavelengths. Cosmologically, there could exist a power deficit in the CMB at the largest scales induced by an early nonequilibrium suppression of Born-rule quantum noise. This possibility has been studied in some detail [23, 24, 26, 27, 28]. We have been unable to predict the current lengthscale for such effects, since this depends on unknown details of the early expansion (such as the number of inflationary e-folds). But we are able to predict the ‘shape’ of the deficit. Recent detailed simulations have shown that the nonequilibrium deficit will have an approximately inverse-tangent dependence on wavenumber  $k$  [28], a prediction which may be compared with current data [44].

In this paper it will be shown that we are also able to predict associated statistical anisotropies and certain features thereof. By a straightforward argument, the form of the anisotropic spectrum follows (in the limit of weak anisotropy) from the inverse-tangent deficit function which has already been obtained from numerical simulations. We then have a single physical mechanism – quantum relaxation on expanding space – that makes detailed predictions for both kinds of anomaly. As we shall see, the mechanism predicts that the statistical anisotropies will be related to the power deficit in quantitative ways which may in principle be testable. Furthermore, the mechanism can naturally induce residual statistical anisotropies at small scales.

In Section 2 we outline the essential formalism required for a discussion of anisotropic primordial power spectra and we provide a brief summary of the statistical anomalies in the CMB. In Section 3 we first summarise the evidence that cosmological quantum relaxation predicts a large-scale power deficit that varies as an inverse-tangent with wavenumber  $k$ . We then show how to generalise this result to an anisotropic power spectrum, where in pilot-wave field theory nonequilibrium anisotropy can exist even in the (conventionally isotropic) Bunch-Davies vacuum. In the limit of weak anisotropy, we derive an expression for the anisotropic primordial power spectrum which is determined by the

inverse-tangent power deficit. In Section 4 we express the CMB covariance matrix (for the harmonic coefficients of the temperature anisotropy) in terms of our anisotropic primordial power spectrum and we show how the lowest-order off-diagonal corrections are determined by our inverse-tangent deficit function. In Section 5 we discuss some anisotropic signatures of cosmological quantum relaxation. We show that the anisotropic correlations between neighbouring CMB multipoles, as quantified by the  $l - (l+1)$  terms in the covariance matrix, scale in a particular way with  $l$  – where the scaling depends on the inverse-tangent form of the power deficit function. Furthermore, we show that our anisotropic power spectrum predicts a residual statistical anisotropy even at small scales. We also derive an approximate consistency relation between the scaling of the  $l - (l+1)$  covariance matrix elements and the scaling of the angular power spectrum  $C_l$  in the limit of high  $l$ . Finally, we briefly consider how cosmological quantum relaxation might account for the anomalous alignment observed between multipoles at very low  $l$ . In Section 6 we summarise our results, compare them with related work, and draw our conclusions.

## 2 Statistical anomalies in the cosmic microwave background

In this section we first briefly summarise some essential formalism. Then we consider the role played by isotropic primordial power spectra and how the assumption of isotropy may be dropped. Finally, we outline the apparent statistical anomalies in the CMB as they are currently understood.

### 2.1 Primordial fluctuations and the CMB

A primordial curvature perturbation  $\mathcal{R}_{\mathbf{k}}$  (working in Fourier space) generates coefficients [45]

$$a_{lm} = \frac{i^l}{2\pi^2} \int d^3\mathbf{k} \mathcal{T}(k, l) \mathcal{R}_{\mathbf{k}} Y_{lm}(\hat{\mathbf{k}}) \quad (1)$$

in the harmonic expansion

$$\frac{\Delta T(\hat{\mathbf{n}})}{\bar{T}} = \sum_{l=2}^{\infty} \sum_{m=-l}^{+l} a_{lm} Y_{lm}(\hat{\mathbf{n}}) \quad (2)$$

of the measured CMB temperature anisotropy  $\Delta T(\hat{\mathbf{n}}) \equiv T(\hat{\mathbf{n}}) - \bar{T}$  (where  $\bar{T}$  is the average temperature over the sky). Here  $\mathcal{T}(k, l)$  is the transfer function and the unit vector  $\hat{\mathbf{n}}$  specifies a point on the sky.

It is usually assumed that the function  $\Delta T(\hat{\mathbf{n}})$  observed by us is a single realisation from an underlying theoretical ensemble with an associated probability distribution. It is common to assume statistical isotropy for the ensemble. This implies that the two-point correlation function  $\langle \Delta T(\hat{\mathbf{n}}) \Delta T(\hat{\mathbf{n}}') \rangle$  – where  $\langle \dots \rangle$  denotes an ensemble average – depends only on the angle between  $\hat{\mathbf{n}}$  and  $\hat{\mathbf{n}}'$ . Using

properties of spherical harmonics, this in turn implies the standard expression [46]

$$\langle a_{l'm'}^* a_{lm} \rangle = \delta_{ll'} \delta_{mm'} C_l \quad (3)$$

for the covariance matrix  $\langle a_{l'm'}^* a_{lm} \rangle$  of the harmonic coefficients, where

$$C_l \equiv \langle |a_{lm}|^2 \rangle \quad (4)$$

is the angular power spectrum.

It is also commonly assumed that the theoretical ensemble for  $\mathcal{R}_{\mathbf{k}}$  is statistically homogeneous. This implies that the real-space two-point correlation function  $\langle \mathcal{R}(\mathbf{x}) \mathcal{R}(\mathbf{x}') \rangle$  depends only on the separation between  $\mathbf{x}$  and  $\mathbf{x}'$ . This in turn implies that  $\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle e^{-i(\mathbf{k}+\mathbf{k}') \cdot \mathbf{d}} = \langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle$  for arbitrary  $\mathbf{d}$ , which implies the standard relation

$$\langle \mathcal{R}_{\mathbf{k}}^* \mathcal{R}_{\mathbf{k}} \rangle = \delta_{\mathbf{k}\mathbf{k}'} \langle |\mathcal{R}_{\mathbf{k}}|^2 \rangle . \quad (5)$$

Using (5) it follows from (1) that

$$C_l = \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k} \mathcal{T}^2(k, l) \mathcal{P}_{\mathcal{R}}(k) , \quad (6)$$

where

$$\mathcal{P}_{\mathcal{R}}(k) \equiv \frac{4\pi k^3}{V} \langle |\mathcal{R}_{\mathbf{k}}|^2 \rangle \quad (7)$$

is the primordial power spectrum (with  $V$  a normalisation volume).

During the inflationary expansion, an inflaton perturbation  $\phi_{\mathbf{k}}$  generates a curvature perturbation  $\mathcal{R}_{\mathbf{k}} \propto \phi_{\mathbf{k}}$ , where  $\phi_{\mathbf{k}}$  is evaluated at a time a few e-folds after the mode exits the Hubble radius [47]. Observational constraints on the  $C_l$ 's imply constraints on  $\mathcal{P}_{\mathcal{R}}(k)$  and hence (since  $\mathcal{R}_{\mathbf{k}} \propto \phi_{\mathbf{k}}$ ) constraints on the primordial variance  $\langle |\phi_{\mathbf{k}}|^2 \rangle$  for  $\phi_{\mathbf{k}}$ . Thus observations of the CMB today imply constraints on quantum fluctuations in the very early universe [48, 49].

In an ideal Bunch-Davies vacuum the inflaton perturbations  $\phi_{\mathbf{k}}$  will have (in the limit  $k/a \ll H$ ) a quantum-theoretical variance

$$\langle |\phi_{\mathbf{k}}|^2 \rangle_{\text{QT}} = \frac{V}{2(2\pi)^3} \frac{H^2}{k^3} , \quad (8)$$

where  $H$  is the (approximately constant) Hubble parameter. This implies a scale-free power spectrum

$$\mathcal{P}_{\phi}^{\text{QT}}(k) \equiv \frac{4\pi k^3}{V} \langle |\phi_{\mathbf{k}}|^2 \rangle_{\text{QT}} = \frac{H^2}{4\pi^2} . \quad (9)$$

The quantity  $\langle |\phi_{\mathbf{k}}|^2 \rangle_{\text{QT}}$  is calculated from quantum field theory. In the slow-roll approximation we then obtain an approximately scale-free quantum-theoretical power spectrum

$$\mathcal{P}_{\mathcal{R}}^{\text{QT}}(k) \equiv \frac{4\pi k^3}{V} \langle |\mathcal{R}_{\mathbf{k}}|^2 \rangle_{\text{QT}} \approx \text{const.} \quad (10)$$

for  $\mathcal{R}_{\mathbf{k}}$ . Of course,  $H$  is only approximately constant during inflation and there will in fact be a mild dependence of  $\mathcal{P}_{\mathcal{R}}^{\text{QT}}(k)$  on  $k$ . This simple treatment suffices for our purposes.

## 2.2 Isotropic and anisotropic primordial power spectra

We have presented the usual justification for the expressions (3) and (6) – assuming that  $\Delta T(\hat{\mathbf{n}})$  is statistically isotropic and that  $\mathcal{R}_{\mathbf{k}}$  is statistically homogeneous. However, (3) and (6) may be derived directly from (1) assuming that  $\langle |\mathcal{R}_{\mathbf{k}}|^2 \rangle$  is independent of the direction of the wave vector  $\mathbf{k}$  (and that  $\mathcal{R}_{\mathbf{k}}$  is statistically homogeneous). This provides a convenient starting point for generalisation, with  $\langle |\mathcal{R}_{\mathbf{k}}|^2 \rangle$  dependent on the direction of  $\mathbf{k}$ .

If  $\mathcal{R}_{\mathbf{k}}$  is statistically homogeneous we have the relations (5) or

$$\langle \mathcal{R}_{\mathbf{k}}^* \mathcal{R}_{\mathbf{k}'} \rangle = \frac{(2\pi)^3}{V} \delta^3(\mathbf{k} - \mathbf{k}') \langle |\mathcal{R}_{\mathbf{k}}|^2 \rangle$$

(using  $V\delta_{\mathbf{k}\mathbf{k}'} = (2\pi)^3\delta^3(\mathbf{k} - \mathbf{k}') \text{ for } V \rightarrow \infty$ ). From (1) we then have

$$\begin{aligned} \langle a_{l'm'}^* a_{lm} \rangle &= \frac{1}{2\pi^2} (-i)^{l'} i^l \int_0^\infty \frac{dk}{k} \mathcal{T}(k, l') \mathcal{T}(k, l) (4\pi k^3/V) \\ &\times \int d\Omega \langle |\mathcal{R}_{\mathbf{k}}|^2 \rangle Y_{l'm'}^*(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{k}}) \end{aligned} \quad (11)$$

(where  $d\Omega$  is a solid-angle element in  $\mathbf{k}$ -space).

If we assume that  $\langle |\mathcal{R}_{\mathbf{k}}|^2 \rangle$  is independent of the direction of  $\mathbf{k}$ , then using the orthonormality property  $\int d\Omega Y_{l'm'}^*(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{k}}) = \delta_{ll'} \delta_{mm'}$  of spherical harmonics we recover the expressions (3) and (6) for the isotropic covariance matrix  $\langle a_{l'm'}^* a_{lm} \rangle$  and for the angular power spectrum  $C_l$  in terms of the isotropic primordial spectrum  $\mathcal{P}_{\mathcal{R}}(k)$ .

If instead we drop the assumption that  $\langle |\mathcal{R}_{\mathbf{k}}|^2 \rangle$  is independent of the direction of  $\mathbf{k}$ , we obtain an anisotropic primordial power spectrum

$$\mathcal{P}_{\mathcal{R}}(k, \hat{\mathbf{k}}) = (4\pi k^3/V) \langle |\mathcal{R}_{\mathbf{k}}|^2 \rangle \quad (12)$$

which depends on the (unit vector) direction  $\hat{\mathbf{k}}$  in  $\mathbf{k}$ -space as well as on the magnitude  $k \equiv |\mathbf{k}|$ . This possibility has been discussed by several authors [50, 51, 52, 53, 17], who have considered power spectra of the form

$$\mathcal{P}_{\mathcal{R}}(k, \hat{\mathbf{k}}) = \mathcal{P}_{\mathcal{R}}(k) \left( 1 + \sum_{LM} g_{LM}(k) Y_{LM}(\hat{\mathbf{k}}) \right), \quad (13)$$

where  $\mathcal{P}_{\mathcal{R}}(k)$  is a standard isotropic spectrum and the functions  $g_{LM}(k)$  quantify scale-dependent anisotropies.

In terms of  $\mathcal{P}_{\mathcal{R}}(k, \hat{\mathbf{k}})$  we have

$$\langle a_{l'm'}^* a_{lm} \rangle = \frac{1}{2\pi^2} (-i)^{l'} i^l \int_0^\infty \frac{dk}{k} \mathcal{T}(k, l') \mathcal{T}(k, l) \int d\Omega \mathcal{P}_{\mathcal{R}}(k, \hat{\mathbf{k}}) Y_{l'm'}^*(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{k}}) . \quad (14)$$

The consequences of an anisotropic spectrum will be explored below.

### 2.3 Apparent anomalies in CMB data

We now briefly consider the various anomalies that appear to exist in the CMB.

Data from the Planck satellite show evidence for an angular power deficit at large scales. The 2013 release reported a deficit of 5–10% in the region  $l \lesssim 40$ , with a statistical significance in the range  $2.5\text{--}3\sigma$  (depending on the estimator used) [54].<sup>1</sup> The more recent 2015 release finds a similar deficit with a slightly reduced significance [1]. A related anomaly has been observed in the (temperature) two-point angular correlation function, which at large scales is measured to be smaller than expected with a statistical significance exceeding  $3\sigma$  [56]. The reported deficit could be a mere statistical fluctuation, or it might indicate a genuine anomaly in the primordial power spectrum – possibly from new physics. In any case, it is important to develop theoretical models that predict a low- $l$  deficit in order to better assess the nature and significance of this finding.

There are also significant hints of statistical anisotropy in various forms and on different scales. For an extensive analysis and survey, see ref. [2]. We briefly summarise the key points.

Signals of statistical anisotropy have been detected in the form of hemispherical asymmetry, point-parity asymmetry (obtained by dividing the temperature anisotropy into spherical harmonics with even and odd  $l$ -modes), mirror-parity asymmetry (obtained from properties of the temperature anisotropy under reflection with respect to a plane), and an improbable alignment between the quadrupole ( $l = 2$ ) and octopole ( $l = 3$ ) modes. This last feature was confirmed in the 2013 Planck data release [57].

Of particular interest to us are forms of statistical anisotropy that manifest as correlations between  $l$  and  $l \pm 1$  multipoles.

There is evidence for a dipolar modulation of the temperature anisotropy, of the form [58]

$$\Delta T(\hat{\mathbf{n}}) = (1 + A \hat{\mathbf{p}} \cdot \hat{\mathbf{n}}) \Delta T_{\text{iso}}(\hat{\mathbf{n}}) , \quad (15)$$

where  $\Delta T_{\text{iso}}$  is the temperature anisotropy of a statistically isotropic sky,  $\hat{\mathbf{p}}$  is a fixed unit vector and the coefficient  $A$  is measured to be  $0.07 \pm 0.02$  [57, 2]. However, significant power in the dipole modulation seems to be limited to  $l \simeq 2 - 64$ ; for  $l \gtrsim 100$  there is no significant detection [57, 2]. The dipolar effects must therefore depend on  $l$ ; the simple temperature modulation (15) is inadequate. One might also consider a possible quadrupolar modulation of the

<sup>1</sup>There were already suggestions that data from the *WMAP* satellite contained anomalously low power at small  $l$ , but such claims were controversial [55].

primordial power spectrum (terms with  $L = 2$  in (13)), which would induce correlations between  $l$  and  $l \pm 2$  CMB multipoles. However, the Planck team finds no evidence for such effects [2].

There is intriguing evidence for statistical anisotropy even at small scales [57, 2]. The nature of the asymmetry is not known, but it appears to be directional in some form. For the 2013 results the relevant multipole range was  $l = 2 - 600$  [57] (confirming earlier findings in the WMAP data [59]). For the 2015 results, the preferred dipolar directions associated with specific multipole ranges appear to be correlated between large and small scales, where the correlations with lower multipoles persist up to  $l = 1500$  [2]. Such correlations are not expected for a statistically isotropic sky.

As emphasised by the Planck team, since many of the anomalies had already been observed in the WMAP data it seems untenable that they could be caused by systematic errors in these two independent experiments.

In this paper we focus on the statistical anisotropy as quantified by  $l - (l \pm 1)$  multipole correlations.

### 3 Unified mechanism for a power deficit with statistical anisotropy

In this section we begin by summarising recent work that has yielded a ‘quantum relaxation spectrum’ given by an inverse-tangent power deficit. We then discuss how this work can be generalised to allow for statistical anisotropy – in the Bunch-Davies vacuum, and in the primordial power spectrum. Finally, we derive an expression for the anisotropic nonequilibrium deficit in the limit of weak anisotropy. In this limit it is possible to derive an explicit result that is fully determined (up to arbitrary constants) by the inverse-tangent power deficit.

#### 3.1 Quantum relaxation spectrum

In de Broglie-Bohm pilot-wave theory [29, 30, 31, 32, 33], a system with wave function  $\psi(q, t)$  has a definite configuration  $q(t)$  whose velocity  $\dot{q} \equiv dq/dt$  is at all times determined by the general formula

$$\frac{dq}{dt} = \frac{j}{|\psi|^2} , \quad (16)$$

where  $j = j[\psi] = j(q, t)$  is the Schrödinger current [60]. The ‘pilot wave’  $\psi(q, t)$  obeys the usual Schrödinger equation  $i\partial\psi/\partial t = \hat{H}\psi$  (with  $\hbar = 1$ ) and is defined in configuration space; it guides the motion of an individual system and has no a priori connection with probability. Unlike in standard quantum theory, an ensemble of systems with the same  $\psi$  can have an arbitrary distribution  $\rho(q, t)$  of configurations. By construction,  $\rho(q, t)$  obeys the continuity equation

$$\frac{\partial\rho}{\partial t} + \partial_q \cdot (\rho\dot{q}) = 0 . \quad (17)$$



The Schrödinger equation for  $\psi$  implies that  $|\psi|^2$  also obeys (17). An initial distribution  $\rho(q, t_i) = |\psi(q, t_i)|^2$  therefore evolves into a final distribution  $\rho(q, t) = |\psi(q, t)|^2$ . This is the state of quantum equilibrium, for which we obtain the Born rule and the usual predictions of quantum theory [31, 32]. For a nonequilibrium ensemble ( $\rho(q, t) \neq |\psi(q, t)|^2$ ), the statistical predictions generally disagree with those of quantum theory [18, 19, 20, 23, 34].

In pilot-wave theory, quantum equilibrium may be understood to arise dynamically. The process of relaxation or equilibration may be quantified by an  $H$ -function  $H = \int dq \rho \ln(\rho/|\psi|^2)$ , which obeys a coarse-graining  $H$ -theorem (where the minimum  $H = 0$  corresponds to equilibrium  $\rho = |\psi|^2$ ) [18, 20, 22]. If  $\psi$  is a superposition of energy eigenstates, numerical simulations show a rapid relaxation  $\rho \rightarrow |\psi|^2$  (on a coarse-grained level) [20, 22, 39, 40, 41, 42, 43], with an approximately exponential decay of the coarse-grained  $H$ -function [39, 41, 43]. It has been suggested that such relaxation took place – at least to a good approximation – in the early universe [18, 19, 20, 21, 22, 23, 24, 26].

For the purposes of this paper, it suffices to consider the pilot-wave dynamics of a free, minimally-coupled, real massless scalar field  $\phi$  on an expanding background with line element  $d\tau^2 = dt^2 - a^2 d\mathbf{x}^2$ . Here  $a = a(t)$  is the scale factor (with  $c = 1$ ). We have a classical Lagrangian density  $\mathcal{L} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ , where  $g_{\mu\nu}$  is the background metric. It is convenient to work with Fourier components  $\phi_{\mathbf{k}} = \frac{\sqrt{V}}{(2\pi)^{3/2}} (q_{\mathbf{k}1} + i q_{\mathbf{k}2})$ , where  $V$  is a normalisation volume and the  $q_{\mathbf{k}r}$ 's ( $r = 1, 2$ ) are real. The field Hamiltonian is then a sum  $H = \sum_{\mathbf{k}r} H_{\mathbf{k}r}$ , where

$$H_{\mathbf{k}r} = \frac{1}{2a^3} \pi_{\mathbf{k}r}^2 + \frac{1}{2} a k^2 q_{\mathbf{k}r}^2. \quad (18)$$

Upon quantisation we obtain the Schrödinger equation

$$i \frac{\partial \Psi}{\partial t} = \sum_{\mathbf{k}r} \left( -\frac{1}{2a^3} \frac{\partial^2}{\partial q_{\mathbf{k}r}^2} + \frac{1}{2} a k^2 q_{\mathbf{k}r}^2 \right) \Psi \quad (19)$$

for the wave functional  $\Psi = \Psi[q_{\mathbf{k}r}, t]$ . We may then identify the de Broglie velocities

$$\frac{dq_{\mathbf{k}r}}{dt} = \frac{1}{a^3} \text{Im} \frac{1}{\Psi} \frac{\partial \Psi}{\partial q_{\mathbf{k}r}} \quad (20)$$

for the configuration degrees of freedom  $q_{\mathbf{k}r}$  [23, 24, 26]. (This construction assumes a preferred foliation of spacetime with time function  $t$  and may be readily generalised to any globally-hyperbolic spacetime [61, 36, 62].)

We may consider the independent dynamics of an unentangled mode  $\mathbf{k}$  with wave function  $\psi_{\mathbf{k}}(q_{\mathbf{k}1}, q_{\mathbf{k}2}, t)$ . Dropping the index  $\mathbf{k}$ , we then have the Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = \sum_{r=1, 2} \left( -\frac{1}{2a^3} \partial_r^2 + \frac{1}{2} a k^2 q_r^2 \right) \psi \quad (21)$$

for  $\psi = \psi(q_1, q_2, t)$  and the de Broglie velocities

$$\dot{q}_r = \frac{1}{a^3} \text{Im} \frac{\partial_r \psi}{\psi} \quad (22)$$

for the configuration  $(q_1, q_2)$  (with  $\partial_r \equiv \partial/\partial q_r$ ). The marginal distribution  $\rho = \rho(q_1, q_2, t)$  obeys the continuity equation

$$\frac{\partial \rho}{\partial t} + \sum_{r=1, 2} \partial_r \left( \rho \frac{1}{a^3} \text{Im} \frac{\partial_r \psi}{\psi} \right) = 0 . \quad (23)$$

Mathematically, the system is equivalent to a two-dimensional oscillator with time-dependent mass  $m = a^3$  and time-dependent angular frequency  $\omega = k/a$  [23, 24]. This is in turn equivalent to a standard oscillator – with constant mass and constant angular frequency – but with real time  $t$  replaced by a ‘retarded time’  $t_{\text{ret}}(t, k)$  [27]. Cosmological relaxation for a single field mode may then be studied in terms of a standard oscillator.

Such studies have been carried out extensively for a radiation-dominated expansion ( $a \propto t^{1/2}$ ) [27, 28]. At short (sub-Hubble) wavelengths,  $t_{\text{ret}}(t, k) \rightarrow t$  and we recover the usual time evolution on Minkowski spacetime – with rapid relaxation for a superposition of excited states. At long (super-Hubble) wavelengths,  $t_{\text{ret}}(t, k) \ll t$  and relaxation is retarded (an effect which may also be understood in terms of an upper bound on the mean displacement of the trajectories [24, 62]). Thus, if there was a radiation-dominated pre-inflationary era, then at the beginning of inflation we may expect to find relic nonequilibrium at long wavelengths [26, 27, 28].

No further relaxation takes place during inflation itself, because the de Broglie-Bohm trajectories of the inflaton field are too simple – assuming that the Bunch-Davies vacuum is a good approximation [23, 26]. The Bunch-Davies wave functional is a product  $\Psi_0[q_{\mathbf{k}r}, t] = \prod_{\mathbf{k}r} \psi_{\mathbf{k}r}(q_{\mathbf{k}r}, t)$  of contracting Gaussian packets and the de Broglie equation (20) for the trajectories  $q_{\mathbf{k}r}(t)$  has a simple exact solution. It is found that the time evolution of an arbitrary nonequilibrium distribution  $\rho_{\mathbf{k}r}(q_{\mathbf{k}r}, t)$  shows the same overall contraction as the time evolution of the equilibrium distribution, so that the ratio

$$\xi(k) \equiv \frac{\langle |\phi_{\mathbf{k}}|^2 \rangle}{\langle |\phi_{\mathbf{k}}|^2 \rangle_{\text{QT}}} \quad (24)$$

of the nonequilibrium mean-square  $\langle |\phi_{\mathbf{k}}|^2 \rangle$  to the quantum-theoretical mean-square  $\langle |\phi_{\mathbf{k}}|^2 \rangle_{\text{QT}}$  is preserved in time.

Thus, in the approximation of a Bunch-Davies vacuum, if relic nonequilibrium ( $\xi \neq 1$ ) exists at the beginning of inflation it will be preserved during the inflationary era and be transferred to larger lengthscales as physical wavelengths  $\lambda_{\text{phys}} = a\lambda = a(2\pi/k)$  expand. The spectrum of primordial curvature perturbations is then expected to take the form [23, 26]

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}_{\mathcal{R}}^{\text{QT}}(k) \xi(k) , \quad (25)$$

where  $\mathcal{P}_{\mathcal{R}}^{\text{QT}}(k)$  is the quantum-theoretical (equilibrium) spectrum. The function  $\xi(k)$  quantifies the degree of nonequilibrium as a function of  $k$ .

From measurements of the angular power spectrum  $C_l$ , we may set observational bounds on  $\xi(k)$  [26]. Of greater interest is to make predictions for  $\xi(k)$  and to compare these with data. To this end, for a single mode in a putative pre-inflationary era we considered initial wave functions that are superpositions

$$\psi(q_1, q_2, t_i) = \frac{1}{\sqrt{M}} \sum_{n_1=0}^{\sqrt{M}-1} \sum_{n_2=0}^{\sqrt{M}-1} e^{i\theta_{n_1 n_2}} \Phi_{n_1}(q_1) \Phi_{n_2}(q_2) \quad (26)$$

of  $M$  energy eigenstates  $\Phi_{n_1} \Phi_{n_2}$  of the initial Hamiltonian, with coefficients  $c_{n_1 n_2}(t_i) = (1/\sqrt{M})e^{i\theta_{n_1 n_2}}$  of equal amplitude and with randomly-chosen phases  $\theta_{n_1 n_2}$ . The wave function at time  $t$  is then

$$\psi(q_1, q_2, t) = \frac{1}{\sqrt{M}} \sum_{n_1=0}^{\sqrt{M}-1} \sum_{n_2=0}^{\sqrt{M}-1} e^{i\theta_{n_1 n_2}} \psi_{n_1}(q_1, t) \psi_{n_2}(q_2, t) , \quad (27)$$

where  $\psi_{n_r}(q_r, t)$  is the time evolution of the initial wave function  $\psi_{n_r}(q_r, t_i) = \Phi_{n_r}(q_r)$  under a formal one-dimensional Schrödinger equation with Hamiltonian  $H_r(t)$ . (The exact solution for  $\psi_n(q, t)$  with  $a \propto t^{1/2}$  was found in ref. [27].) At time  $t$  the equilibrium distribution is  $\rho_{\text{QT}}(q_1, q_2, t) = |\psi(q_1, q_2, t)|^2$ . In the simulations reported in refs. [27, 28], we assumed for simplicity an initial nonequilibrium distribution

$$\rho(q_1, q_2, t_i) = |\Phi_0(q_1) \Phi_0(q_2)|^2 = \frac{\omega_i m_i}{\pi} e^{-m_i \omega_i q_1^2} e^{-m_i \omega_i q_2^2} . \quad (28)$$

(This is just the equilibrium distribution for the ground state  $\Phi_0(q_1) \Phi_0(q_2)$ .) The initial nonequilibrium width is smaller than the initial equilibrium width. We calculated numerically the time evolution  $\rho(q_1, q_2, t)$  of the ensemble distribution up to a final time  $t_f$ . This was done for varying values of  $k$ , as well as for varying values of  $M$  and  $t_f$ . In each case we compared the final nonequilibrium width (that is, the width of  $\rho(q_1, q_2, t_f)$ ) with the final equilibrium width (that is, the width of  $|\psi(q_1, q_2, t_f)|^2$ ) – thereby yielding values of the ratio  $\xi$  for varying values of  $k$ ,  $M$  and  $t_f$ . (Here  $\xi$  is defined as the ratio of the nonequilibrium and equilibrium variances and not of the mean-squares as in (24) which holds only when the means vanish. Furthermore, before taking their ratio, the respective variances were averaged over mixtures of initial wave functions (26) with randomly-chosen phases. See ref. [28] for details.)

For given  $M$  and  $t_f$ , extensive numerical simulations for varying  $k$  have yielded functions  $\xi(k)$  that take the form of an inverse-tangent [28]. Specifically, in terms of the convenient variable

$$x = 2k/H_0 , \quad (29)$$

the numerical results show good fits to the curve

$$\xi(k) = \tan^{-1}(\tilde{c}_1 x + c_2) - \frac{\pi}{2} + c_3 , \quad (30)$$

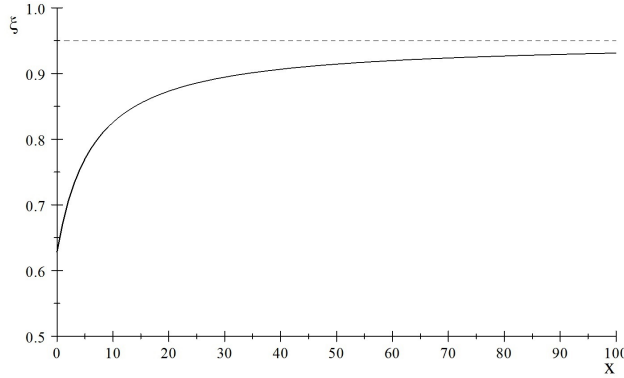


Figure 1: Plot of the illustrative power deficit function  $\xi = \tan^{-1}(0.5x+3) - \frac{\pi}{2} + 0.95$  (with  $x = 2k/H_0$ ). The dashed line shows the asymptotic value  $c_3 = 0.95$ .

where  $\tilde{c}_1$ ,  $c_2$ ,  $c_3$  are constant parameters that depend on  $M$  and  $t_f$ . Here  $\tilde{c}_1 = c_1 R H_0 / 20\pi$ , where the constant  $c_1$  is obtained from our simulations of a pre-inflationary era and the factor  $R \equiv a(t_{\text{today}})/a(t_f)$  is required to rescale our wave numbers taking into account the spatial expansion from the end of pre-inflation at time  $t_f$  until today at time  $t_{\text{today}}$  [28]. (Our numerical results for  $c_1$ ,  $c_2$ ,  $c_3$  are listed in tables I and II of ref. [28].) Note that for large  $x$  we have  $\xi \rightarrow c_3$ .

In ref. [28] we considered  $(\tilde{c}_1, c_2) \simeq (0.5, 2.85)$  as an illustrative example of values that are consistent with both our numerical simulations and the approximate magnitude of the observed angular power deficit, as well as with basic cosmological constraints on  $R$ . For definiteness, as ‘fiducial’ or illustrative values we may simply take

$$(\tilde{c}_1, c_2) = (0.5, 3) \quad (31)$$

(as will be used below). For  $c_3$ , in one case with  $M = 12$  we found  $c_3 = 0.95$ . Taking this value for illustration, together with (31), in Figure 1 we plot the resulting  $\xi$  as a function of  $x$ .

The numerical results for  $\xi(k)$  actually show oscillations around the curve (30) of magnitude  $\lesssim 10\%$  (see figure 3 of ref. [28]). To a first approximation we ignore the oscillations, whose contribution we hope to consider in future work.

The overall shape of the curve (30) shows the retardation or suppression of relaxation at long wavelengths, with  $\xi(k)$  decreasing for smaller  $k$ . At short wavelengths, or large  $k$ , we might expect complete relaxation – in which case we should have  $\xi \rightarrow 1$ . Instead, we have  $\xi \rightarrow c_3$  and the numerical results show that  $c_3$  can differ from 1. Such a nonequilibrium ‘residue’ at short wavelengths is equivalent to an overall renormalisation of the power spectrum and is therefore by itself observationally indistinguishable from a compensating shift in other cosmological parameters [28, 44].

The numerical results of ref. [28] indicate that the large- $k$  residue  $c_3$  approaches 1 for large  $M$ . As expected, we find equilibrium to good accuracy ( $\xi \simeq 1$ ) in a limiting regime with both short wavelengths and large numbers of modes. However, for small  $M$  the residue  $c_3$  is not quite equal to 1. Even at very short wavelengths, for small  $M$  relaxation is unlikely to occur completely because the trajectories are unlikely to fully explore the configuration space (depending on the initial phases in the wavefunction) [43]. In our scenario there is a preference for small  $M$  during pre-inflation, because after the transition to inflation a significant number of excitations above the vacuum are likely to cause a back-reaction problem [63]. Thus, in a realistic working scenario  $c_3$  is likely to differ slightly from 1.

The aim of ref. [28] was to obtain a general signature of cosmological quantum relaxation that is as far as possible independent of details of the pre-inflationary era. In more recent work [64], the generality of the inverse-tangent result (30) has been extended in two ways. First, we have studied initial nonequilibrium distributions that are more complicated than the initial ‘vacuum’ Gaussian (28). Second, we have included a period of exponential expansion after the radiation-dominated phase. In both cases, we still obtain a function  $\xi(k)$  of the form (30). We may therefore reasonably expect a deficit of the form (30) in any scenario involving cosmological quantum relaxation. Note that while we usually find  $c_3 < 1$ , sometimes we find  $c_3 > 1$  [64].

In all of our numerical simulations of quantum relaxation, we assume initial distributions  $\rho(q_1, q_2, t_i)$  whose width is smaller than the initial equilibrium width (that is, smaller than the width of  $|\psi(q_1, q_2, t_i)|^2$ ). In principle, this assumption could be violated by the actual initial conditions of our universe. However, we make this assumption partly for simplicity and partly on heuristic grounds: if quantum noise has a dynamical origin, it seems natural to assume initial conditions with a subquantum statistical spread (so that the initial state contains less noise than an ordinary quantum state). In any case, some assumptions need to be made about the initial conditions in order to make any predictions at all. Another assumption we make is that  $\rho(q_1, q_2, t_i)$  does not possess any fine-grained micro-structure; that is, we assume that  $\rho(q_1, q_2, t_i)$  is approximately constant over the size of our coarse-graining cells (used to calculate the decreasing coarse-grained  $H$ -function). Some such assumption is always required in order to obtain relaxation in the statistical mechanics of a time-reversal invariant theory.<sup>2</sup> Our assumptions about the initial state are guided by simplicity and heuristic arguments, but in the end their justification rests on how well the resulting predictions compare with observation.

In the reported numerical studies, the scalar field  $\phi$  is regarded as representative of whatever generic fields may have been present in a pre-inflationary era. The relation between our field  $\phi$  and the later inflaton field is not specified. This is a significant gap in our proposed scenario, which remains to be filled. We simply assume that the simulated deficit function  $\xi(k)$  at the end of pre-inflation more or less survives the transition to inflation in the sense that the

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<sup>2</sup>For discussions of this point in pilot-wave theory, see refs. [20, 21, 22, 39].

same correction appears in the inflationary spectrum [26, 27, 28]. A more complete understanding must await the development of a model of the transition from pre-inflation to inflation – which may for example involve symmetry breaking and associated field redefinitions. It is conceivable that the transition will in fact significantly change the functional form of the predicted deficit. Pending a proper treatment, we assume that this is not the case. With this caveat, we may say that the deficit function (30) is a robust prediction of cosmological quantum relaxation.

In ref. [28] we also studied relaxation for the phases of our primordial scalar field components  $\phi_{\mathbf{k}}$ , with a view to understanding statistical anisotropy. We found that anomalous phases are likely to exist at scales comparable to that of the power deficit. However, the connection with the observed anisotropy of the CMB was not clearly established as there is no simple relationship between the phases of primordial perturbations  $\mathcal{R}_{\mathbf{k}} \propto \phi_{\mathbf{k}}$  and the phases of CMB harmonic coefficients  $a_{lm}$ . Here we follow a simpler and more direct route to statistical anisotropy by considering primordial perturbations whose variance depends on the wave vector direction  $\hat{\mathbf{k}}$  (a suggestion that was in fact briefly made at the end of ref. [27]).

### 3.2 Statistical anisotropy in the Bunch-Davies vacuum

Let us consider how statistical anisotropy can exist in the Bunch-Davies vacuum. In conventional quantum field theory this would be a contradiction in terms, but it is perfectly possible in pilot-wave field theory. To show this, we first recall some basic properties of the Bunch-Davies vacuum  $|0\rangle$  for a free scalar field  $\phi$  on de Sitter spacetime ( $a \propto e^{Ht}$ ) [23, 26].

The Bunch-Davies wave functional  $\Psi_0[q_{\mathbf{k}r}, t] = \langle q_{\mathbf{k}r} | 0 \rangle$  may be calculated from the defining equation  $\hat{a}_{\mathbf{k}} | 0 \rangle = 0$ , where  $\hat{a}_{\mathbf{k}}$  is an appropriate annihilation operator [26]. The result is a product  $\Psi_0[q_{\mathbf{k}r}, t] = \prod_{\mathbf{k}r} \psi_{\mathbf{k}r}(q_{\mathbf{k}r}, t)$  of wave functions  $\psi_{\mathbf{k}r} = |\psi_{\mathbf{k}r}| e^{is_{\mathbf{k}r}}$ , where

$$|\psi_{\mathbf{k}r}|^2 = \frac{1}{\sqrt{2\pi\Delta_k^2}} e^{-q_{\mathbf{k}r}^2/2\Delta_k^2} \quad (32)$$

is a Gaussian of contracting squared width

$$\Delta_k^2 = \frac{H^2}{2k^3} \left( 1 + \frac{k^2}{H^2 a^2} \right) \quad (33)$$

and the phase  $s_{\mathbf{k}r}$  is given by

$$s_{\mathbf{k}r} = -\frac{ak^2 q_{\mathbf{k}r}^2}{2H(1 + k^2/H^2 a^2)} + h(t) , \quad (34)$$

where

$$h(t) = \frac{1}{2} \left( \frac{k}{Ha} - \tan^{-1} \left( \frac{k}{Ha} \right) \right) . \quad (35)$$

The wave function  $\psi_{\mathbf{k}r} = |\psi_{\mathbf{k}r}| e^{i s_{\mathbf{k}r}}$  satisfies the Schrödinger equation for a single mode  $\mathbf{k}r$ , and in the limit  $H \rightarrow 0$ ,  $a \rightarrow 1$  reduces to the wave function of the Minkowski vacuum  $\psi_{\mathbf{k}r} \propto e^{-\frac{1}{2} k q_{\mathbf{k}r}^2} e^{-i \frac{1}{2} k t}$ . Note that the width  $\Delta_{\mathbf{k}r} = \Delta_k$  depends on the magnitude  $k = |\mathbf{k}|$  of the wave vector but not on its direction  $\hat{\mathbf{k}}$ .

In terms of conformal time  $\eta = -1/Ha$  (ranging from  $-\infty$  to 0), the de Broglie equation of motion for  $q_{\mathbf{k}r}$  reads

$$\frac{dq_{\mathbf{k}r}}{d\eta} = \frac{k^2 \eta q_{\mathbf{k}r}}{1 + k^2 \eta^2}, \quad (36)$$

with the solution [23, 26]

$$q_{\mathbf{k}r}(\eta) = q_{\mathbf{k}r}(0) \sqrt{1 + k^2 \eta^2} \quad (37)$$

(in terms of the asymptotic values at  $\eta = 0$ ). It is also easy to obtain the time evolution of an arbitrary initial probability distribution. For simplicity we may assume that the initial distribution  $P[q_{\mathbf{k}r}, \eta_i]$  factorises across modes. For a single mode, the marginal distribution  $\rho_{\mathbf{k}r}(q_{\mathbf{k}r}, \eta)$  (generally  $\neq |\psi_{\mathbf{k}r}(q_{\mathbf{k}r}, \eta)|^2$ ) satisfies the continuity equation

$$\frac{\partial \rho_{\mathbf{k}r}}{\partial \eta} + \frac{\partial}{\partial q_{\mathbf{k}r}} \left( \rho_{\mathbf{k}r} \frac{dq_{\mathbf{k}r}}{d\eta} \right) = 0 \quad (38)$$

(with the velocity field (36)). This has the general solution [26]

$$\rho_{\mathbf{k}r}(q_{\mathbf{k}r}, \eta) = \frac{1}{\sqrt{1 + k^2 \eta^2}} \rho_{\mathbf{k}r}(q_{\mathbf{k}r}/\sqrt{1 + k^2 \eta^2}, 0) \quad (39)$$

for any given  $\rho_{\mathbf{k}r}(q_{\mathbf{k}r}, 0)$  (again writing formally in terms of asymptotic values at  $\eta = 0$ ).

At times  $\eta < 0$ ,  $|\psi_{\mathbf{k}r}|^2$  is a contracting Gaussian of width

$$\Delta_k(\eta) = \Delta_k(0) \sqrt{1 + k^2 \eta^2} \quad (40)$$

(with  $\Delta_k(0) = \sqrt{H^2/2k^3}$ ) while  $\rho_{\mathbf{k}r}$  is a contracting arbitrary distribution of width

$$D_{\mathbf{k}r}(\eta) = D_{\mathbf{k}r}(0) \sqrt{1 + k^2 \eta^2} \quad (41)$$

(with arbitrary  $D_{\mathbf{k}r}(0)$ ). Both distributions contract by the same overall factor and so their widths remain in a constant ratio.

Now, in earlier studies [23, 26, 27] it was assumed for simplicity that  $D_{\mathbf{k}r}(\eta) = D_k(\eta)$ . The nonequilibrium width  $D_{\mathbf{k}r}$  was taken to be independent of the direction of  $\mathbf{k}$  (as well as independent of  $r$ ), as is always the case for the equilibrium width  $\Delta_{\mathbf{k}r} = \Delta_k$ . We then defined the deficit function as the ratio  $\xi(k) = D_k^2/\Delta_k^2$ , which reduces to (24) when the distributions have zero mean. (The mean is necessarily zero for the equilibrium vacuum, and for simplicity we assume it to be zero for the nonequilibrium vacuum as well.) However, for a

general nonequilibrium state there is no physical reason – other than simplicity – for why the width  $D_{\mathbf{k}r}$  should not also depend on the wave vector direction  $\hat{\mathbf{k}}$  (as well as on  $r$ ).

We may then define a direction-dependent deficit function  $\xi = \xi(k, \hat{\mathbf{k}})$  by the (time-independent) ratio

$$\xi(k, \hat{\mathbf{k}}) \equiv \frac{D_{\mathbf{k}1}^2 + D_{\mathbf{k}2}^2}{\Delta_{\mathbf{k}1}^2 + \Delta_{\mathbf{k}2}^2}, \quad (42)$$

where  $D_{\mathbf{k}r}^2 = \langle q_{\mathbf{k}r}^2 \rangle - \langle q_{\mathbf{k}r} \rangle^2$  and  $\Delta_{\mathbf{k}r}^2 = \langle q_{\mathbf{k}r}^2 \rangle_{\text{QT}} - \langle q_{\mathbf{k}r} \rangle_{\text{QT}}^2$ . The equilibrium variance  $\Delta_{\mathbf{k}r}^2 = \Delta_k^2$  is in fact independent of  $\hat{\mathbf{k}}$  and  $r$ . We also have  $\langle q_{\mathbf{k}r} \rangle_{\text{QT}} = 0$ . If we assume that the nonequilibrium mean also vanishes,  $\langle q_{\mathbf{k}r} \rangle = 0$ , we then have

$$\xi(k, \hat{\mathbf{k}}) = \frac{\langle q_{\mathbf{k}1}^2 \rangle + \langle q_{\mathbf{k}2}^2 \rangle}{\langle q_{\mathbf{k}1}^2 \rangle_{\text{QT}} + \langle q_{\mathbf{k}2}^2 \rangle_{\text{QT}}} = \frac{\langle |\phi_{\mathbf{k}}|^2 \rangle}{\langle |\phi_{\mathbf{k}}|^2 \rangle_{\text{QT}}}. \quad (43)$$

Thus our direction-dependent mean-square perturbation may be written as

$$\langle |\phi_{\mathbf{k}}|^2 \rangle = \langle |\phi_{\mathbf{k}}|^2 \rangle_{\text{QT}} \xi(k, \hat{\mathbf{k}}). \quad (44)$$

Statistical isotropy is violated – even though the wave functional itself is still that of the Bunch-Davies vacuum.

### 3.3 Derivation of the anisotropic deficit function

If we take the field perturbations  $\phi_{\mathbf{k}}$  to generate primordial curvature perturbations  $\mathcal{R}_{\mathbf{k}} \propto \phi_{\mathbf{k}}$ , then from (44) we will have a direction-dependent mean-square

$$\langle |\mathcal{R}_{\mathbf{k}}|^2 \rangle = \langle |\mathcal{R}_{\mathbf{k}}|^2 \rangle_{\text{QT}} \xi(k, \hat{\mathbf{k}}) \quad (45)$$

and an anisotropic primordial power spectrum

$$\mathcal{P}_{\mathcal{R}}(k, \hat{\mathbf{k}}) = \mathcal{P}_{\mathcal{R}}^{\text{QT}}(k) \xi(k, \hat{\mathbf{k}}). \quad (46)$$

The isotropic equilibrium spectrum  $\mathcal{P}_{\mathcal{R}}^{\text{QT}}(k)$  is modulated by the anisotropic deficit function  $\xi(k, \hat{\mathbf{k}})$ .

What could be the physical origin of such an anisotropic deficit? The equilibrium spectrum  $\mathcal{P}_{\mathcal{R}}^{\text{QT}}(k)$  is necessarily isotropic because the quantum width  $\Delta_{\mathbf{k}r} = \Delta_k$  of inflationary vacuum modes is independent of  $\hat{\mathbf{k}}$ . However, if quantum nonequilibrium existed in a pre-inflationary phase we may expect it to be anisotropic since the nonequilibrium width  $D_{\mathbf{k}r}$  of pre-inflationary field modes will generally depend on  $\hat{\mathbf{k}}$ . Furthermore, relaxation suppression at long wavelengths will preserve the anisotropy (as well as any power deficit) for small  $k$  while efficient relaxation at short wavelengths will drive the field modes close to the quantum isotropic spectrum (with the usual power) for large  $k$ . Thus the dynamics of pilot-wave field theory on expanding space provides a natural



mechanism whereby both a power deficit and a statistical anisotropy can be generated at large scales while approximately yielding the standard predictions at small scales. In fact, as we shall see, at small scales we expect to find an approximately scale-free spectrum but with residual statistical anisotropies. It is this possibility – of a unified explanation for the large-scale deficit and the large-scale anisotropy, together with the small-scale anisotropy – that we wish to explore and develop in this paper.

A general anisotropic power spectrum  $\mathcal{P}_{\mathcal{R}}(k, \hat{\mathbf{k}})$  may be written in the form (13) with

$$\xi(k, \hat{\mathbf{k}}) = 1 + \sum_{LM} g_{LM}(k) Y_{LM}(\hat{\mathbf{k}}) , \quad (47)$$

where  $g_{LM}(k)$  are scale-dependent harmonic coefficients in  $\mathbf{k}$ -space. In a general phenomenology, the  $g_{LM}(k)$  would be arbitrary functions of  $k$ . However, our model with cosmological quantum relaxation implies a specific  $k$ -dependence (in the limit of weak anisotropy).

The isotropic nonequilibrium deficit function (30) was obtained from numerical simulations [28]. These were carried out for single field modes of variable wave number  $k$  and fixed wave vector direction  $\hat{\mathbf{k}}$ . For each  $k$  an average was taken over an ensemble of wave functions with randomly-chosen initial phases. While the dynamics depends on  $k$ , it is blind to the direction  $\hat{\mathbf{k}}$ . We may then imagine carrying out a generalised simulation for single field modes of variable  $k$  and  $\hat{\mathbf{k}}$ , where the initial conditions – in particular the initial nonequilibrium distribution  $\rho(q_{\mathbf{k}1}, q_{\mathbf{k}2}, t_i)$  and its width – are allowed to depend on  $\hat{\mathbf{k}}$ . For each mode with given  $k$  and  $\hat{\mathbf{k}}$ , we again take an average over an ensemble of wave functions with randomly-chosen initial phases. Now different initial nonequilibrium states will in general yield different coefficients  $\tilde{c}_1, c_2, c_3$  in the function  $\xi(k)$  generated by the simulations. (Some examples are in fact found in ref. [64].) Thus, if field modes with different wave vector directions  $\hat{\mathbf{k}}$  have different initial nonequilibrium distributions and widths, then in effect the generated coefficients  $\tilde{c}_1, c_2, c_3$  will depend on  $\hat{\mathbf{k}}$  and we may consider functions  $\tilde{c}_1(\hat{\mathbf{k}}), c_2(\hat{\mathbf{k}}), c_3(\hat{\mathbf{k}})$ . Let us write these as

$$\tilde{c}_1(\hat{\mathbf{k}}) = \tilde{c}_1 + \Delta_1(\hat{\mathbf{k}}) , \quad c_2(\hat{\mathbf{k}}) = c_2 + \Delta_2(\hat{\mathbf{k}}) , \quad c_3(\hat{\mathbf{k}}) = c_3 + \Delta_3(\hat{\mathbf{k}}) ,$$

with the  $\Delta_i(\hat{\mathbf{k}})$  expanded as

$$\Delta_i(\hat{\mathbf{k}}) = \sum_{L \geq 1, M} \Delta_{LM}^i Y_{LM}(\hat{\mathbf{k}}) \quad (48)$$

(for  $i = 1, 2, 3$ , where the coefficients  $\Delta_{LM}^i$  are independent of  $k$ ). In this way the isotropic function (30) is generalised to the anisotropic function

$$\xi(k, \hat{\mathbf{k}}) = \tan^{-1} \left[ \left( \tilde{c}_1 + \Delta_1(\hat{\mathbf{k}}) \right) x + c_2 + \Delta_2(\hat{\mathbf{k}}) \right] - \frac{\pi}{2} + c_3 + \Delta_3(\hat{\mathbf{k}}) .$$

In the limit of weak anisotropy we may take the  $\Delta_i(\hat{\mathbf{k}})$  to be small. Expanding to lowest order in  $\Delta_i(\hat{\mathbf{k}})$  and defining

$$f_1(x) \equiv \frac{\partial \xi(k)}{\partial \tilde{c}_1} , \quad f_2(x) \equiv \frac{\partial \xi(k)}{\partial c_2} , \quad f_3(x) \equiv \frac{\partial \xi(k)}{\partial c_3} \quad (49)$$

(again with  $x = 2k/H_0$ ), we have

$$\xi(k, \hat{\mathbf{k}}) = \xi(k) + \sum_{i=1}^3 f_i(x) \Delta_i(\hat{\mathbf{k}}) , \quad (50)$$

where from (30) we find the functions

$$f_1(x) = \frac{x}{1 + (\tilde{c}_1 x + c_2)^2} , \quad f_2(x) = \frac{1}{1 + (\tilde{c}_1 x + c_2)^2} , \quad f_3(x) = 1 . \quad (51)$$

The anisotropic part of  $\xi(k, \hat{\mathbf{k}})$  depends on  $\tilde{c}_1$  and  $c_2$  but is independent of  $c_3$ . Below we shall consider the functions  $f_i(x)$  with the illustrative values (31) for  $\tilde{c}_1, c_2$ .

Using the expansions (48), we obtain an expression for the anisotropic deficit function

$$\xi(k, \hat{\mathbf{k}}) = \xi(k) + \sum_{L \geq 1, M} \left( \sum_{i=1}^3 f_i(x) \Delta_{LM}^i \right) Y_{LM}(\hat{\mathbf{k}}) \quad (52)$$

in terms of unknown coefficients  $\Delta_{LM}^i$  but with known scaling functions  $f_i(x)$ .

The result (52) may be compared with the general expression (47). We have  $Y_{00} = 1/\sqrt{4\pi}$  and so

$$g_{00}(k) = -\sqrt{4\pi} (1 - \xi(k)) . \quad (53)$$

For  $L \geq 1$  we have

$$g_{LM}(k) = \sum_{i=1}^3 f_i(x) \Delta_{LM}^i . \quad (54)$$

The third term ( $i = 3$ ) is independent of  $k$ , whereas the first two terms scale with  $k$  and tend to zero for large  $k$ .

Note that for (52) we have  $\frac{1}{4\pi} \int d\Omega \xi(k, \hat{\mathbf{k}}) = \xi(k)$ . The angular average of  $\xi(k, \hat{\mathbf{k}})$  is equal to the average isotropic result  $\xi(k)$  obtained from the numerical simulations.

We may now explore the implications of the anisotropic deficit function (50) or (52).

In the expansions (48) the reality of  $\Delta_i(\hat{\mathbf{k}})$  implies that

$$(\Delta_{LM}^i)^* = (-1)^M \Delta_{L(-M)}^i , \quad (55)$$

where  $Y_{LM}^* = (-1)^M Y_{L(-M)}$ . Taking angular coordinates  $(\theta, \phi)$  in  $\mathbf{k}$ -space, with  $Y_{10} = \sqrt{3/4\pi} \cos \theta$  and  $Y_{11} = -\sqrt{3/8\pi} \sin \theta e^{i\phi}$ , the first few terms in the expansions take the form

$$\Delta_i(\hat{\mathbf{k}}) = \Delta_{10}^i \sqrt{3/4\pi} \cos \theta - \Delta_{11}^i \sqrt{3/8\pi} e^{i\phi} \sin \theta - (\Delta_{11}^i)^* \sqrt{3/8\pi} e^{-i\phi} \sin \theta + \dots , \quad (56)$$

where the  $\Delta_{10}^i$  are real.

We might reasonably assume that the first few terms will be the most important. For simplicity we shall consider only the lowest ( $L = 1$ ) terms. As discussed further in Section 6, in our scenario the initial anisotropy is in principle arbitrary and it is arguably natural to expect the lowest ( $L = 1$ ) terms to dominate.

## 4 Anisotropic corrections to the covariance matrix

In Section 2.3 we expressed the covariance matrix  $\langle a_{l'm'}^* a_{lm} \rangle$  in terms of a general anisotropic power spectrum  $\mathcal{P}_{\mathcal{R}}(k, \hat{\mathbf{k}})$ . The result (14) is valid as long as the primordial perturbations  $\mathcal{R}_{\mathbf{k}}$  are statistically homogeneous (as we assume throughout). For  $\mathcal{P}_{\mathcal{R}}(k, \hat{\mathbf{k}}) = \mathcal{P}_{\mathcal{R}}^{\text{QT}}(k) \xi(k, \hat{\mathbf{k}})$  we then have

$$\langle a_{l'm'}^* a_{lm} \rangle = \frac{(-i)^{l'} i^l}{2\pi^2} \int_0^\infty \frac{dk}{k} \mathcal{T}(k, l') \mathcal{T}(k, l) \mathcal{P}_{\mathcal{R}}^{\text{QT}}(k) \chi_{l'm'lm}(k) , \quad (57)$$

where we have defined

$$\chi_{l'm'lm}(k) \equiv \int d\Omega \xi(k, \hat{\mathbf{k}}) Y_{l'm'}^*(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{k}}) . \quad (58)$$

In quantum equilibrium  $\xi(k, \hat{\mathbf{k}}) = 1$  and the quantities  $\chi_{l'm'lm}(k)$  take the quantum-theoretical values

$$\chi_{l'm'lm}^{\text{QT}}(k) = \delta_{ll'} \delta_{mm'} . \quad (59)$$

We then recover the equilibrium covariance matrix

$$\langle a_{l'm'}^* a_{lm} \rangle_{\text{QT}} = \delta_{ll'} \delta_{mm'} C_l^{\text{QT}} \quad (60)$$

and the equilibrium angular power spectrum

$$C_l^{\text{QT}} = \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k} \mathcal{T}^2(k, l) \mathcal{P}_{\mathcal{R}}^{\text{QT}}(k) \quad (61)$$

in terms of the (isotropic) equilibrium primordial spectrum  $\mathcal{P}_{\mathcal{R}}^{\text{QT}}(k)$ . The expressions (60) and (61) are usually associated with fundamental statistical isotropy. Here we regard them as peculiar to quantum equilibrium – which happens to yield statistical isotropy in the standard inflationary vacuum.

If instead  $\xi(k, \hat{\mathbf{k}})$  depends on the direction vector  $\hat{\mathbf{k}}$ , then isotropy is generally broken and the nonequilibrium covariance matrix (57) takes a more general form.

### 4.1 Covariance matrix with the anisotropic deficit function

Let us consider our anisotropic deficit function in the general form (50), for which the quantities (58) become

$$\chi_{l'm'lm}(k) \equiv \xi(k) \delta_{ll'} \delta_{mm'} + \sum_{i=1}^3 f_i(x) I_i(l', m'; l, m)$$

where

$$I_i(l', m'; l, m) \equiv \int d\Omega \Delta_i(\hat{\mathbf{k}}) Y_{l'm'}^*(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{k}}) . \quad (62)$$

Because  $\Delta_i$  is real we must have

$$I_i^*(l', m'; l, m) = I_i(l, m; l', m') . \quad (63)$$

Our nonequilibrium covariance matrix (57) then becomes

$$\langle a_{l'm'}^* a_{lm} \rangle = \delta_{ll'} \delta_{mm'} C_l + \frac{(-i)^{l'} i^l}{2\pi^2} \sum_{i=1}^3 I_i(l', m'; l, m) F_i(l', l) \quad (64)$$

where  $C_l$  is now the corrected angular power spectrum

$$C_l = \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k} \mathcal{T}^2(k, l) \mathcal{P}_{\mathcal{R}}^{\text{QT}}(k) \xi(k) \quad (65)$$

(written in terms of an isotropic primordial spectrum  $\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}_{\mathcal{R}}^{\text{QT}}(k) \xi(k)$  with an isotropic deficit function  $\xi(k)$ ), and where for  $i = 1, 2, 3$  we have defined the integrals

$$F_i(l', l) \equiv \int_0^\infty \frac{dk}{k} \mathcal{T}(k, l') \mathcal{T}(k, l) \mathcal{P}_{\mathcal{R}}^{\text{QT}}(k) f_i(x) . \quad (66)$$

Note that the anisotropic part of  $\langle a_{l'm'}^* a_{lm} \rangle$  depends on  $\tilde{c}_1$  and  $c_2$  but is independent of  $c_3$ .

We are particularly interested in the off-diagonal matrix elements. For  $l'm' \neq lm$  we have

$$\langle a_{l'm'}^* a_{lm} \rangle = \frac{(-i)^{l'} i^l}{2\pi^2} \sum_{i=1}^3 I_i(l', m'; l, m) F_i(l', l) . \quad (67)$$

Note that the off-diagonal terms depend on our anisotropic functions  $f_i(x)$  (given by (51)), where these were determined by our isotropic deficit function  $\xi(k)$  via the equations (49). Thus the off-diagonal terms contain information about the form of the function  $\xi(k)$ , which our simulations have constrained to be (approximately) an inverse-tangent.

Considering only the lowest ( $L = 1$ ) terms in the expansions (56) for  $\Delta_i(\hat{\mathbf{k}})$ , we may study the integrals  $I_i(l', m'; l, m)$  and obtain the lowest-order anisotropic (off-diagonal) corrections (67) to the covariance matrix.<sup>3</sup>

## 4.2 Multipole $l - (l + 1)$ correlations

We first evaluate the integrals (62), with  $\hat{\mathbf{k}}$  represented by angular coordinates  $(\theta, \phi)$ . Considering only the lowest ( $L = 1$ ) terms in the expansions (56) for

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<sup>3</sup>In future work we hope to consider these effects in terms of the convenient formalism of bipolar spherical harmonics [46, 65]. The approach taken here suffices for our present purposes.

$\Delta_i(\hat{\mathbf{k}})$ , we have

$$\begin{aligned}
I_i(l', m'; l, m) &= \Delta_{10}^i \sqrt{3/4\pi} \int d\Omega Y_{l'm'}^*(\theta, \phi) \cos \theta Y_{lm}(\theta, \phi) \\
&\quad - \Delta_{11}^i \sqrt{3/8\pi} \int d\Omega Y_{l'm'}^*(\theta, \phi) e^{i\phi} \sin \theta Y_{lm}(\theta, \phi) \\
&\quad - (\Delta_{11}^i)^* \sqrt{3/8\pi} \int d\Omega Y_{l'm'}^*(\theta, \phi) e^{-i\phi} \sin \theta Y_{lm}(\theta, \phi) .
\end{aligned} \tag{68}$$

We may use the following relations (ref. [66], p. 805):

$$\begin{aligned}
\cos \theta Y_{lm}(\theta, \phi) &= \alpha_1 Y_{(l+1)m}(\theta, \phi) + \alpha_2 Y_{(l-1)m}(\theta, \phi) , \\
e^{i\phi} \sin \theta Y_{lm}(\theta, \phi) &= -\alpha_3 Y_{(l+1)(m+1)}(\theta, \phi) + \alpha_4 Y_{(l-1)(m+1)}(\theta, \phi) , \\
e^{-i\phi} \sin \theta Y_{lm}(\theta, \phi) &= \alpha_5 Y_{(l+1)(m-1)}(\theta, \phi) - \alpha_6 Y_{(l-1)(m-1)}(\theta, \phi) ,
\end{aligned} \tag{69}$$

where

$$\begin{aligned}
\alpha_1 &= \sqrt{\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}} , \quad \alpha_2 = \sqrt{\frac{(l-m)(l+m)}{(2l-1)(2l+1)}} , \\
\alpha_3 &= \sqrt{\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}} , \quad \alpha_4 = \sqrt{\frac{(l-m)(l-m-1)}{(2l-1)(2l+1)}} , \\
\alpha_5 &= \sqrt{\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}} , \quad \alpha_6 = \sqrt{\frac{(l+m)(l+m-1)}{(2l-1)(2l+1)}} .
\end{aligned} \tag{70}$$

The integrals in (68) then follow trivially from the orthonormality relations  $\int d\Omega Y_{l'm'}^* Y_{lm} = \delta_{l'l} \delta_{m'm}$ . We find

$$\begin{aligned}
I_i(l', m'; l, m) &= \delta_{l'(l+1)} \sqrt{3/4\pi} \left[ \Delta_{10}^i \alpha_1 \delta_{m'm} + \Delta_{11}^i \frac{1}{\sqrt{2}} \alpha_3 \delta_{m'(m+1)} - (\Delta_{11}^i)^* \frac{1}{\sqrt{2}} \alpha_5 \delta_{m'(m-1)} \right] \\
&\quad + \delta_{l'(l-1)} \sqrt{3/4\pi} \left[ \Delta_{10}^i \alpha_2 \delta_{m'm} - \Delta_{11}^i \frac{1}{\sqrt{2}} \alpha_4 \delta_{m'(m+1)} + (\Delta_{11}^i)^* \frac{1}{\sqrt{2}} \alpha_6 \delta_{m'(m-1)} \right] .
\end{aligned} \tag{71}$$

Note that

$$I_i(l, m; l, m) = 0 .$$

Thus for  $l'm' = lm$  we have simply

$$\langle |a_{lm}|^2 \rangle = C_l , \tag{72}$$

where  $C_l$  is the corrected angular power spectrum (65) with deficit function  $\xi(k)$ .

From (71) for  $l'm' \neq lm$ , the only non-zero off-diagonal values of  $I_i(l', m'; l, m)$  are

$$I_i(l+1, m'; l, m) = \sqrt{3/4\pi} \left[ \Delta_{10}^i \alpha_1 \delta_{m'm} + \Delta_{11}^i \frac{1}{\sqrt{2}} \alpha_3 \delta_{m'(m+1)} - (\Delta_{11}^i)^* \frac{1}{\sqrt{2}} \alpha_5 \delta_{m'(m-1)} \right] \tag{73}$$

and

$$I_i(l-1, m'; l, m) = \sqrt{3/4\pi} \left[ \Delta_{10}^i \alpha_2 \delta_{m'm} - \Delta_{11}^i \frac{1}{\sqrt{2}} \alpha_4 \delta_{m'(m+1)} + (\Delta_{11}^i)^* \frac{1}{\sqrt{2}} \alpha_6 \delta_{m'(m-1)} \right]. \quad (74)$$

Thus, from (67), the only non-vanishing off-diagonal correlations  $\langle a_{l'm'}^* a_{lm} \rangle$  are

$$\langle a_{(l+1)m'}^* a_{lm} \rangle = -\frac{i}{2\pi^2} \sum_{i=1}^3 I_i(l+1, m'; l, m) F_i(l+1, l) \quad (75)$$

and

$$\langle a_{(l-1)m'}^* a_{lm} \rangle = \frac{i}{2\pi^2} \sum_{i=1}^3 I_i(l-1, m'; l, m) F_i(l-1, l), \quad (76)$$

where

$$F_i(l \pm 1, l) = \int_0^\infty \frac{dk}{k} \mathcal{T}(k, l \pm 1) \mathcal{T}(k, l) \mathcal{P}_{\mathcal{R}}^{\text{QT}}(k) f_i(x). \quad (77)$$

We emphasise that the off-diagonal terms (75) and (76) in the covariance matrix depend on the form of our isotropic deficit function  $\xi(k)$  (via the associated functions  $f_i(x)$ ).

## 5 Anisotropic signatures of cosmological quantum relaxation

The aim of this paper is to find signatures of cosmological quantum relaxation in statistical CMB anisotropies. In particular, it would be desirable to have signatures that are independent of the unknown – and as far as we know arbitrary – coefficients  $\Delta_{10}^i$  and  $\Delta_{11}^i$  appearing in the anisotropic expansions (56). Such signatures will be presented in this section. Whether or not they might be detectable in practice is outside the scope of this paper, but some relevant comments will be made in the concluding Section 6.

As we have noted, the quantum-theoretical spectrum  $\mathcal{P}_{\mathcal{R}}^{\text{QT}}(k)$  is necessarily isotropic because the quantum equilibrium width of inflationary vacuum modes is independent of the direction  $\hat{\mathbf{k}}$  of the wave vector  $\mathbf{k}$ . On the other hand, early quantum nonequilibrium is expected to be anisotropic: the nonequilibrium width of the relevant field modes can generally depend on  $\hat{\mathbf{k}}$ . In our quantum relaxation scenario, we expect to find residual nonequilibrium at long wavelengths and (at least) approximate equilibrium at short wavelengths – where the transition from one regime to another is precisely characterised by our inverse-tangent deficit function  $\xi(k)$  (equation (30)) obtained from numerical simulations. As the wavenumber  $k$  increases, equilibrium is approached and the anisotropic nonequilibrium width approaches (at least approximately) the isotropic equilibrium width. Thus anisotropy at large scales necessarily gives way to approximate isotropy at small scales.

Now because we have a precise characterisation of how the width deficit  $\xi(k)$  varies with  $k$ , in the limit of weak anisotropy we are able to deduce how the

anisotropies in the generalised deficit  $\xi(k, \hat{\mathbf{k}})$  vary with  $k$  (via the argument given in Section 3.3). As a result, we are able to make predictions for the behaviour of the anisotropies in the covariance matrix as a function of multipole moment  $l$ . Furthermore, we are able to predict residual statistical anisotropies at small scales.

These results amount to constraints, or consistency relations, between the impact of quantum nonequilibrium on the power deficit and the impact of quantum nonequilibrium on statistical anisotropy. In our scenario, the two kinds of anomaly are closely related and we are in fact able to deduce quantitative relationships between them. In principle, the deduced relationships could be tested against CMB data (though we do not attempt to do so here).

Our scenario is also likely to have implications for the question of anomalous mode alignment at large scales. This is discussed briefly in Section 5.4.

### 5.1 Low- $l$ scaling of $l - (l + 1)$ correlations with $l$

Let us study how our correlation matrix elements  $\langle a_{(l+1)m'}^* a_{lm} \rangle$  – as given by (75) – scale as a function of multipole moment  $l$  in the low- $l$  regime. (The other non-zero elements  $\langle a_{(l-1)m'}^* a_{lm} \rangle$  are trivially equal to  $\langle a_{lm}^* a_{(l-1)m'} \rangle^*$ .)

We wish to extract an approximate scaling with  $l$  that is independent of the unknown coefficients  $\Delta_{10}^i$  and  $\Delta_{11}^i$  in the expansions (56). To this end, we consider the average over  $m'$  and  $m$ :

$$\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle} \equiv \frac{1}{2(l+1)+1} \frac{1}{2l+1} \sum_{m'=-l}^{l+1} \sum_{m=-l}^l \langle a_{(l+1)m'}^* a_{lm} \rangle. \quad (78)$$

From (75) this may be written as

$$\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle} = -\frac{i}{2\pi^2} \sum_{i=1}^3 \overline{I_i(l+1, m'; l, m)} F_i(l+1, l), \quad (79)$$

where

$$\overline{I_i(l+1, m'; l, m)} \equiv \frac{1}{2(l+1)+1} \frac{1}{2l+1} \sum_{m'=-l}^{l+1} \sum_{m=-l}^l I_i(l+1, m'; l, m). \quad (80)$$

Let us consider the integral factors in (79). As a simple approximation we may take  $\mathcal{P}_{\mathcal{R}}^{\text{QT}}(k) \simeq \text{const.} \equiv \mathcal{P}_{\mathcal{R}}^{\text{QT}}$ . At low  $l$  the transfer function takes the form [47]

$$\mathcal{T}(k, l) = \sqrt{\pi} H_0^2 j_l(2k/H_0) \quad (81)$$

(where  $H_0$  is the Hubble parameter today). We take (81) to be valid for (say)  $l \lesssim 20$  – ignoring the small contribution from the integrated Sachs-Wolfe effect at very low  $l$ . In this range of  $l$  we then have

$$F_i(l+1, l) \simeq \pi H_0^4 \mathcal{P}_{\mathcal{R}}^{\text{QT}} h_i(l) \quad (82)$$

with

$$h_i(l) \equiv \int_0^\infty \frac{dx}{x} j_{l+1}(x) j_l(x) f_i(x) , \quad (83)$$

where again it is convenient to work with the parameter  $x = 2k/H_0$ .

Our averaged covariance matrix elements (79) then take the form

$$\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle} = -\frac{i}{2\pi} H_0^4 \mathcal{P}_{\mathcal{R}}^{\text{QT}} \sum_{i=1}^3 \overline{I_i(l+1, m'; l, m)} h_i(l) . \quad (84)$$

The dependence on  $l$  is contained in the factors  $\overline{I_i(l+1, m'; l, m)}$  and  $h_i(l)$ . Let us study these in turn.

From (73) we have

$$\overline{I_i(l+1, m'; l, m)} = \frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{1}{2l+3} \frac{1}{2l+1} \left( \Delta_{10}^i s_1 + \Delta_{11}^i \frac{1}{\sqrt{2}} s_3 - (\Delta_{11}^i)^* \frac{1}{\sqrt{2}} s_5 \right) \quad (85)$$

where

$$s_1 \equiv \sum_{m=-l}^l \alpha_1(l, m) , \quad s_3 \equiv \sum_{m=-l}^l \alpha_3(l, m) , \quad s_5 \equiv \sum_{m=-l}^l \alpha_5(l, m) .$$

Writing

$$s_n = \frac{1}{\sqrt{(2l+1)(2l+3)}} \sigma_n(l)$$

(for  $n = 1, 3, 5$ ) and using (70), in the region  $l = 2$  to  $l = 20$  we find by numerical evaluation that

$$\begin{aligned} \sigma_1(l) &= \sum_{m=-l}^l \sqrt{(l-m+1)(l+m+1)} \approx 2.51 \times l^{15/8} , \\ \sigma_3(l) &= \sum_{m=-l}^l \sqrt{(l+m+1)(l+m+2)} \approx 3.22 \times l^{15/8} . \end{aligned}$$

Also,

$$\sigma_5(l) = \sum_{m=-l}^l \sqrt{(l-m+1)(l-m+2)} = \sum_{m=-l}^l \sqrt{(l+m+1)(l+m+2)} = \sigma_3(l) .$$

For all three terms we have  $\sigma_n(l) \propto l^{15/8}$  (to a good approximation). If we take  $1/\sqrt{(2l+1)(2l+3)} \sim 1/l$  – which is fairly accurate except for the very lowest  $l$  – we find the same approximate scaling

$$s_n \propto l^{7/8} \quad (86)$$



for all three terms appearing in (85). Taking  $\frac{1}{2l+3} \frac{1}{2l+1} \sim 1/l^2$  – again fairly accurate except for the very lowest  $l$  – we then have  $\overline{I_i(l+1, m'; l, m)} \sim l^{-9/8}$  or simply

$$\overline{I_i(l+1, m'; l, m)} \sim l^{-1} \quad (87)$$

(where  $l^{-9/8}$  agrees closely with  $0.85l^{-1}$  in this range of  $l$ ).

Let us now study the factors  $h_i(l)$ . Taking the illustrative values (31) for  $\tilde{c}_1, c_2$ , we have

$$f_1(x) = \frac{x}{1 + (0.5x + 3)^2}, \quad f_2(x) = \frac{1}{1 + (0.5x + 3)^2}, \quad f_3(x) = 1, \quad (88)$$

and

$$h_1(l) = \int_0^\infty \frac{dx}{x} j_{l+1}(x) j_l(x) \frac{x}{1 + (0.5x + 3)^2}, \quad (89)$$

$$h_2(l) = \int_0^\infty \frac{dx}{x} j_{l+1}(x) j_l(x) \frac{1}{1 + (0.5x + 3)^2}, \quad (90)$$

$$h_3(l) = \int_0^\infty \frac{dx}{x} j_{l+1}(x) j_l(x). \quad (91)$$

The integrals (89), (90), (91) may be evaluated numerically.<sup>4</sup> Results for  $l = 2, 3, 4, \dots, 20$  are displayed in Figure 2. We find close fits to the curves

$$h_1(l) = 0.018 \times l^{-3/2}, \quad h_2(l) = 0.0065 \times l^{-2}, \quad h_3(l) = 0.12 \times l^{-3/2}. \quad (92)$$

We may conclude that  $h_1, h_3 \propto l^{-3/2}$  and  $h_2 \propto l^{-2}$  to a good approximation (in this low- $l$  region). These scalings are peculiar to our inverse-tangent deficit function (30) (cf. Section 5.2).

Putting our results (87) and (92) together, we find that different terms in (84) can scale with two different powers of  $l$ . For  $i = 1, 3$  we have terms that scale as  $\sim l^{-1} \times l^{-3/2} = l^{-5/2}$ , while for  $i = 2$  we have terms that scale as  $\sim l^{-1} \times l^{-2} = l^{-3}$ . Thus the terms in  $\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle}$  will scale as

$$\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle} \sim l^{-5/2}, \quad l^{-3}. \quad (93)$$

We conclude that, for our inverse-tangent deficit function (30), the averaged  $l - (l+1)$  covariance matrix elements (84) will contain terms that scale approximately as  $l^{-5/2}$  or as  $l^{-3}$  at low  $l$  – regardless of the values of  $\Delta_{10}^i, \Delta_{11}^i$  in the expansions (56).

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<sup>4</sup>I am grateful to Samuel Colin for assistance with the numerical evaluation of these integrals.

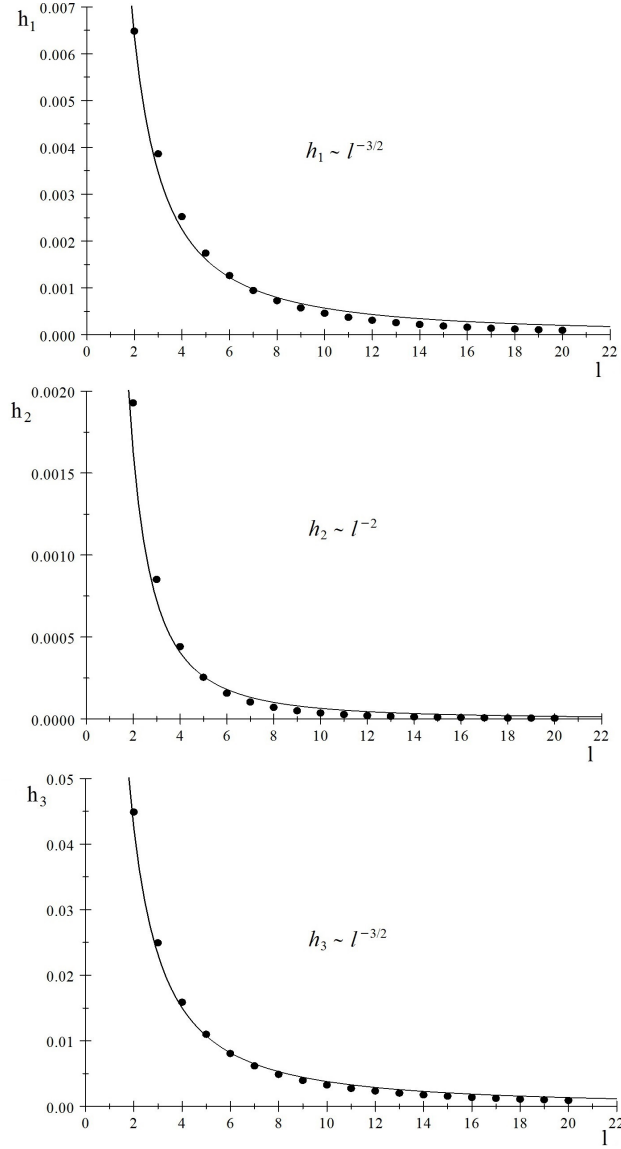


Figure 2: Numerical results for the integrals (89), (90), (91) in the range  $l = 2 - 20$ , with fits to the respective curves  $h_1 \propto l^{-3/2}$ ,  $h_2 \propto l^{-2}$ ,  $h_3 \propto l^{-3/2}$ .

## 5.2 Low- $l$ correlation scaling with an alternative deficit function

The scaling of  $\overline{\langle a_{(l+1)m}^* a_{lm} \rangle}$  with  $l$  depends on the form of the deficit function  $\xi(k)$ . We may illustrate this by considering an alternative deficit function.

We take a three-parameter function  $\xi_{\text{dip}}(k)$  with near-equilibrium ( $\xi_{\text{dip}} = \kappa_3$  where  $\kappa_3$  is understood to be close to 1) for  $x > \kappa_2$  and a sudden dip ( $\xi_{\text{dip}} = \kappa_1 < \kappa_3$ ) in power for  $x < \kappa_2$ :

$$\begin{aligned}\xi_{\text{dip}}(k) &= \kappa_1 \quad \text{for } x < \kappa_2, \\ \xi_{\text{dip}}(k) &= \kappa_3 \quad \text{for } x > \kappa_2.\end{aligned}$$

We may write this as

$$\xi_{\text{dip}}(k) = \kappa_1 \theta(\kappa_2 - x) + \kappa_3 \theta(x - \kappa_2). \quad (94)$$

Let us follow the same procedure as before, generalising the coefficients  $\kappa_i$  ( $i = 1, 2, 3$ ) to direction-dependent functions  $\kappa_i(\hat{\mathbf{k}})$ , which we write as  $\kappa_i(\hat{\mathbf{k}}) = \kappa_i + \Delta_i(\hat{\mathbf{k}})$  with  $\Delta_i(\hat{\mathbf{k}})$  assumed small and expanded as in (48). Our alternative isotropic function (94) is then generalised to the alternative anisotropic function

$$\xi_{\text{dip}}(k, \hat{\mathbf{k}}) = \left( \kappa_1 + \Delta_1(\hat{\mathbf{k}}) \right) \theta(\kappa_2 + \Delta_2(\hat{\mathbf{k}}) - x) + \left( \kappa_3 + \Delta_3(\hat{\mathbf{k}}) \right) \theta(x - \kappa_2 - \Delta_2(\hat{\mathbf{k}})). \quad (95)$$

Expanding to lowest order in  $\Delta_i(\hat{\mathbf{k}})$  and defining the alternative functions

$$f_1^{\text{dip}}(x) \equiv \frac{\partial \xi_{\text{dip}}(k)}{\partial \kappa_1}, \quad f_2^{\text{dip}}(x) \equiv \frac{\partial \xi_{\text{dip}}(k)}{\partial \kappa_2}, \quad f_3^{\text{dip}}(x) \equiv \frac{\partial \xi_{\text{dip}}(k)}{\partial \kappa_3}, \quad (96)$$

we have

$$\xi_{\text{dip}}(k, \hat{\mathbf{k}}) = \xi_{\text{dip}}(k) + \sum_{i=1}^3 f_i^{\text{dip}}(x) \Delta_i(\hat{\mathbf{k}}) \quad (97)$$

where

$$f_1^{\text{dip}}(x) = \theta(\kappa_2 - x), \quad f_2^{\text{dip}}(x) = (\kappa_1 - \kappa_3) \delta(x - \kappa_2), \quad f_3^{\text{dip}}(x) = \theta(x - \kappa_2). \quad (98)$$

In this case the anisotropic part of  $\xi_{\text{dip}}(k, \hat{\mathbf{k}})$  depends on all three parameters  $\kappa_1, \kappa_2, \kappa_3$ .

Note that, despite the appearance of the delta-function  $\delta(x - \kappa_2)$  in  $f_2^{\text{dip}}(x)$ , for small  $\Delta_2(\hat{\mathbf{k}})$  the right-hand side of (97) is a good approximation to the left-hand side (to first order in  $\Delta_2(\hat{\mathbf{k}})$ ) when both sides appear under an integral  $\int dx$  with a function that varies little over a distance  $\Delta_2(\hat{\mathbf{k}})$ . In our case, where  $\xi_{\text{dip}}(k, \hat{\mathbf{k}})$  appears under the integral (57) for  $\langle a_{l'm}^* a_{lm} \rangle$ , at low  $l$  we require that  $\Delta_2(\hat{\mathbf{k}})$  be small compared with the scale over which  $j_{l+1}(x)j_l(x)$  varies with  $x$ .

Let us consider a specific alternative function (94) that is numerically not far from the inverse-tangent function (30) with the illustrative values (31) for

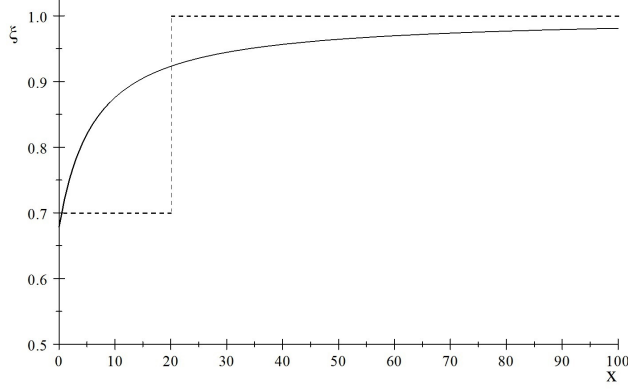


Figure 3: The deficit function  $\xi = \tan^{-1}(0.5x + 3) - \frac{\pi}{2} + 1$  (solid line, with  $x = 2k/H_0$ ) arising from cosmological quantum relaxation and the alternative function  $\xi_{\text{dip}}$  (dashed line, given by (99)) corresponding to a sudden dip in power.

$\tilde{c}_1$ ,  $c_2$  and (for simplicity) with  $c_3 = 1$ . We take

$$\begin{aligned} \xi_{\text{dip}}(k) &= 0.7 \quad \text{for } x < 20, \\ \xi_{\text{dip}}(k) &= 1 \quad \text{for } x > 20. \end{aligned} \quad (99)$$

(The area under the curve  $1 - \xi_{\text{dip}}(k)$  is then equal to 6, which approximately matches the value of  $\int_0^\infty (1 - \xi(k))dx$ . See Figure 3.) This corresponds to a deficit function  $\xi_{\text{dip}}(k)$  with equilibrium ( $\xi_{\text{dip}} = 1$ ) for  $x > 20$  and a sudden dip ( $\xi_{\text{dip}} = 0.7$ ) in power for  $x < 20$  – that is, a dip for  $\lambda > (\pi/5)H_0^{-1}$ . In terms of our parameters, we have

$$\kappa_1 = 0.7, \quad \kappa_2 = 20, \quad \kappa_3 = 1. \quad (100)$$

Our alternative ‘fiducial’ functions are then

$$f_1^{\text{dip}}(x) = \theta(20 - x), \quad f_2^{\text{dip}}(x) = -0.3\delta(x - 20), \quad f_3^{\text{dip}}(x) = \theta(x - 20). \quad (101)$$

We then have the alternative integrals:

$$h_1^{\text{dip}}(l) = \int_0^\infty \frac{dx}{x} j_{l+1}(x) j_l(x) \theta(20 - x) = \int_0^{20} \frac{dx}{x} j_{l+1}(x) j_l(x), \quad (102)$$

$$h_2^{\text{dip}}(l) = \int_0^\infty \frac{dx}{x} j_{l+1}(x) j_l(x) [-0.3\delta(x - 20)] = -\frac{3}{200} j_{l+1}(20) j_l(20), \quad (103)$$

$$h_3^{\text{dip}}(l) = \int_0^\infty \frac{dx}{x} j_{l+1}(x) j_l(x) \theta(x - 20) = \int_{20}^\infty \frac{dx}{x} j_{l+1}(x) j_l(x). \quad (104)$$

These may again be evaluated numerically. Results for  $l = 2, 3, 4, \dots, 20$  are displayed in Figure 4. For  $h_1^{\text{dip}}(l)$  and  $h_3^{\text{dip}}(l)$  we find (respectively close and approximate) fits to the curves

$$h_1^{\text{dip}}(l) = 0.12 \times l^{-3/2}, \quad h_3^{\text{dip}}(l) = 7 \times 10^{-5} + 1.85 \times 10^{-6} l^2, \quad (105)$$

where in the latter case there are small oscillations around the fitted curve. We may conclude that  $h_1^{\text{dip}} \sim l^{-3/2}$  and  $h_3^{\text{dip}} \sim \text{const.} + l^2$  (in this low- $l$  region). For  $h_2^{\text{dip}}(l)$ , there are strong oscillations that grow with  $l$ . These simple scalings for  $h_1^{\text{dip}}(l)$  and  $h_3^{\text{dip}}(l)$ , together with the oscillatory behaviour of  $h_2^{\text{dip}}(l)$ , are peculiar to the alternative deficit function (94).

The averaged  $l - (l+1)$  covariance matrix elements now take the form (from (84))

$$\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle}_{\text{dip}} = -\frac{i}{2\pi} H_0^4 \mathcal{P}_{\mathcal{R}}^{\text{QT}} \sum_{i=1}^3 \overline{I_i(l+1, m'; l, m)} h_i^{\text{dip}}(l),$$

where we still have  $\overline{I_i(l+1, m'; l, m)} \sim l^{-1}$  (equation (87)). Thus for  $i = 1$  we have terms in  $\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle}_{\text{dip}}$  that scale as  $\sim l^{-1} \times l^{-3/2} = l^{-5/2}$  (just as in the case of an inverse-tangent deficit). For  $i = 2$  we have terms that oscillate strongly with  $l$ . Finally, for  $i = 3$  we have terms that scale as  $\sim l^{-1} \times (\text{const.} + l^2)$ ; that is, we have terms that scale as  $\sim l^{-1}$  and  $\sim l$ . Thus there will be terms in  $\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle}_{\text{dip}}$  that scale as

$$\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle}_{\text{dip}} \sim l^{-5/2}, l^{-1}, l \quad (106)$$

(as well as terms that oscillate strongly with  $l$ ).

We note the contrast with the results in Section 5.1 for the inverse-tangent deficit, where we found that  $\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle} \sim l^{-5/2}, l^{-3}$ . The scaling  $\sim l^{-3}$  distinguishes the inverse-tangent case from the dip, while the scalings  $\sim l^{-1}, l$  distinguish the case of a dip from the inverse-tangent. If at low  $l$  the observed averaged matrix elements  $\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle}$  were found to contain terms that scale as  $\sim l^{-3}$ , this would be inconsistent with a sudden dip in primordial power but consistent with our inverse-tangent deficit. On the other hand if the observations found terms that scale as  $\sim l^{-1}$  or  $\sim l$ , this would be inconsistent with our inverse-tangent deficit but consistent with a sudden dip.

We emphasise that these results are independent of the values of the coefficients  $\Delta_{10}^i, \Delta_{11}^i$  in the expansions (56). They reflect properties of the isotropic power deficit function  $\xi(k)$ . The two examples studied here – the inverse-tangent and the sudden dip – suffice to illustrate how the scaling of  $\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle}$  with  $l$  depends on the form of  $\xi(k)$ . It would be of interest to consider other deficit functions, such as power laws, and to study the scaling of  $\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle}$  with  $l$  for different cases. We leave this for future work.

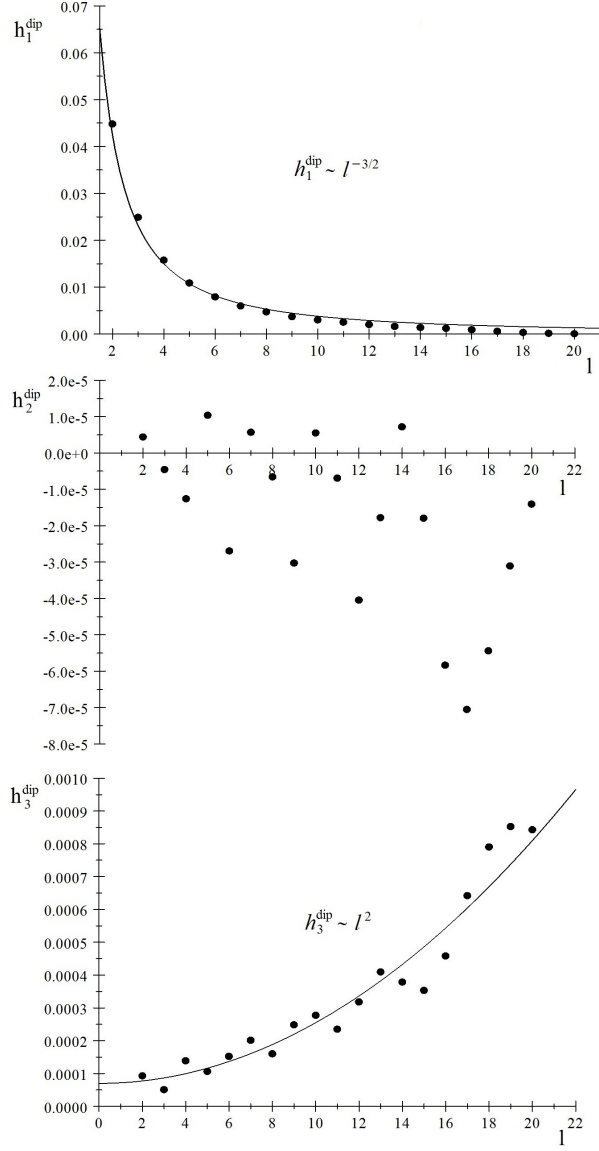


Figure 4: Numerical results for the integrals (102), (103), (104) in the range  $l = 2 - 20$ , with fitted curves  $h_1^{\text{dip}} \sim l^{-3/2}$  and  $h_3^{\text{dip}} \sim \text{const.} + l^2$ .

### 5.3 Residual anisotropies at high $l$

Our simulations of cosmological quantum relaxation generally yield  $c_3$  slightly unequal to 1, so that there is a small nonequilibrium residue in the large- $k$  limit. We then expect to find residual statistical anisotropy even in the limit of high  $l$ . As we noted in Section 2.3, there is evidence in the data for statistical anisotropy at high  $l$ , together with anomalous correlations between directional asymmetries at large and small scales [2].

Let us then consider our anisotropic deficit function (50) at small scales. In the large- $k$  limit we have

$$\xi(k) \simeq c_3, \quad f_1(x) \simeq 0, \quad f_2(x) \simeq 0,$$

while for all  $k$  we have  $f_3(x) = 1$  exactly. Thus for large  $k$  we have simply

$$\xi(k, \hat{\mathbf{k}}) \simeq c_3 + \Delta_3(\hat{\mathbf{k}}) = c_3 + \sum_{L \geq 1, M} \Delta_{LM}^3 Y_{LM}(\hat{\mathbf{k}}) \quad (107)$$

(with  $\Delta_{LM}^3$  presumed small). In terms of the expansion (47), our coefficients (53) and (54) read

$$g_{00}(k) = -\sqrt{4\pi}(1 - c_3)$$

and (for  $L \geq 1$ )

$$g_{LM}(k) = \Delta_{LM}^3.$$

The large- $k$  expression (107) indeed implies that small statistical anisotropies will still be present even at the smallest lengthscales. In fact, at very small lengthscales the anisotropies are scale-independent.

Let us consider the covariance matrix  $\langle a_{l'm'}^* a_{lm} \rangle$  at small scales. Using the full anisotropic deficit function (50), and again taking only the lowest terms in the expansions (56) of  $\Delta_i(\hat{\mathbf{k}})$ , we have the general results (75), (76) for the only non-vanishing off-diagonal covariance matrix elements  $\langle a_{(l \pm 1)m'}^* a_{lm} \rangle$ , with the factors  $I_i(l \pm 1, m'; l, m)$  given by (73), (74) and the factors  $F_i(l \pm 1, l)$  given by (77). To calculate  $\langle a_{(l \pm 1)m'}^* a_{lm} \rangle$  at high  $l$ , we would need to use the transfer function  $\mathcal{T}(k, l)$  at high  $l$ . We leave a detailed analysis for future work. Here we simply note that, in the limit of large  $l$ , if we write  $\mathcal{T}(k, l+1)\mathcal{T}(k, l) \simeq \mathcal{T}^2(k, l)$  then

$$F_i(l+1, l) \simeq F_i(l, l) = \int_0^\infty \frac{dk}{k} \mathcal{T}^2(k, l) \mathcal{P}_{\mathcal{R}}^{\text{QT}}(k) f_i(x) \quad (108)$$

and (75) becomes

$$\langle a_{(l+1)m'}^* a_{lm} \rangle \simeq -\frac{i}{2\pi^2} \sum_{i=1}^3 I_i(l+1, m'; l, m) F_i(l, l) \quad (109)$$

(where again the other non-zero elements  $\langle a_{(l-1)m'}^* a_{lm} \rangle$  are trivially equal to  $\langle a_{lm}^* a_{(l-1)m'} \rangle^*$ ).

Let us consider the factors  $F_i(l, l)$ . For  $i = 3$  we have  $f_3(x) = 1$  and so (using (61))

$$F_3(l, l) = \int_0^\infty \frac{dk}{k} \mathcal{T}^2(k, l) \mathcal{P}_{\mathcal{R}}^{\text{QT}}(k) = 2\pi^2 C_l^{\text{QT}} .$$

We may relate this to the corrected  $C_l$  as given by (65). We are interested in the behaviour at large  $l$ . If we assume that in this regime the integral in (65) is dominated by the large- $k$  limit of the integrand – for which  $\xi(k) \simeq c_3$  – we obtain

$$C_l \simeq c_3 C_l^{\text{QT}} .$$

We then have the simple result

$$F_3(l, l) \simeq (2\pi^2/c_3) C_l .$$

For the terms with  $i = 1, 2$ , if again we assume that for large  $l$  the integral in (108) is dominated by the large- $k$  limit of the integrand – for which now  $f_1(x) \simeq 0$  and  $f_2(x) \simeq 0$  – we then have

$$F_1(l, l) \simeq 0 , \quad F_2(l, l) \simeq 0 .$$

In this approximation, then, from (109) we have simply (for large  $l$ )

$$\left\langle a_{(l+1)m'}^* a_{lm} \right\rangle \simeq -\frac{i}{2\pi^2} I_3(l+1, m'; l, m) (2\pi^2/c_3) C_l . \quad (110)$$

The approximate result (110) amounts to a consistency relation between the off-diagonal covariance matrix elements  $\left\langle a_{(l+1)m'}^* a_{lm} \right\rangle$  and the angular power spectrum  $C_l$  at high  $l$ . Of course the factor  $I_3(l+1, m'; l, m)$  contains the unknown anisotropy coefficients  $\Delta_{10}^3$  and  $\Delta_{11}^3$ , but even so (110) relates the scaling of  $\left\langle a_{(l+1)m'}^* a_{lm} \right\rangle$  with  $l$  to the scaling of  $C_l$  with  $l$ .

Let us consider the factor  $I_3(l+1, m'; l, m)$  for large  $l$ . From the expressions (70), we find (for  $l \gg 1$ )

$$\alpha_1 \simeq \frac{1}{2} \sqrt{1 - m^2/l^2}, \quad \alpha_3 \simeq \frac{1}{2} (1 + m/l), \quad \alpha_5 \simeq \frac{1}{2} (1 - m/l) .$$

From (73) we then have

$$\begin{aligned} I_3(l+1, m'; l, m) \simeq & \frac{1}{2} \sqrt{3/4\pi} [\Delta_{10}^3 \sqrt{1 - m^2/l^2} \delta_{m'm} + \\ & \Delta_{11}^3 (1/\sqrt{2}) (1 + m/l) \delta_{m'(m+1)} - (\Delta_{11}^3)^* (1/\sqrt{2}) (1 - m/l) \delta_{m'(m-1)}] . \end{aligned} \quad (111)$$

The dependence on  $l$  is negligible for  $m \ll l$ .

We may also consider the average (80) of  $I_3(l+1, m'; l, m)$  over  $m$  and  $m'$  (for large  $l$ ). We find

$$\overline{I_3(l+1, m'; l, m)} \simeq \frac{1}{16} \sqrt{\frac{3}{2\pi}} \frac{1}{l^2} \left[ \Delta_{10}^3 \sqrt{2} (1 + 1.58l) + \Delta_{11}^3 (2l+1) - (\Delta_{11}^3)^* (2l+1) \right] \quad (112)$$



(where  $\sum_{m=-l}^l \sqrt{1 - m^2/l^2} \simeq 1 + 1.58l$ ). Thus  $\overline{I_3(l+1, m'; l, m)}$  contains terms that scale as  $l^{-2}$  and  $l^{-1}$ . We then have an  $m$ -averaged consistency relation

$$\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle} \simeq -\frac{i}{2\pi^2} \overline{I_3(l+1, m'; l, m)} (2\pi^2/c_3) C_l. \quad (113)$$

This implies that, at high  $l$ , the  $m$ -averaged covariance matrix will contain terms that scale as

$$\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle} \sim C_l/l^2, \quad C_l/l. \quad (114)$$

The small-scale anisotropies are related to the large-scale anisotropies in the sense that the same (unknown) coefficients  $\Delta_{10}^3$  and  $\Delta_{11}^3$  appear in the small-scale  $l - (l+1)$  correlations (110) as appear in the term  $i = 3$  of the large-scale  $l - (l+1)$  correlations (75) (and the same is true after  $m$ -averaging, as in (113) and (84)). Furthermore, the small-scale correlations (110) depend on the residue coefficient  $c_3$ , which can be obtained by fitting the deficit function (30) to the current data for the power deficit (via the relation (65) between the corrected  $C_l$ 's and  $\xi(k)$ ).<sup>5</sup>

The relationship between small-scale and large-scale anisotropies could be tested observationally, at least in principle. If we could measure or constrain the large-scale covariance matrix elements  $\langle a_{(l+1)m'}^* a_{lm} \rangle$  to sufficient accuracy, then from the expression (75) we could deduce the values of the coefficients  $\Delta_{10}^3$  and  $\Delta_{11}^3$  in the term for  $i = 3$  (where the function  $F_3(l+1, l)$  is known). We could then compare with the coefficients  $\Delta_{10}^3$  and  $\Delta_{11}^3$  appearing in the expression (110) for the small-scale matrix elements  $\langle a_{(l+1)m'}^* a_{lm} \rangle$  – having already found  $c_3$  from a best-fit of  $\xi(k)$  to the power deficit. Clearly, there is a relationship between the  $l - (l+1)$  correlations at large and small scales regardless of the actual values of  $\Delta_{10}^3$  and  $\Delta_{11}^3$ . Similar reasoning applies to the  $m$ -averaged matrix elements  $\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle}$ . Whether or not this has any bearing on the reported correlations between directional asymmetries at large and small scales remains to be seen.

## 5.4 Mode alignment

As we mentioned in Section 2.3, in their 2013 data release the Planck team confirmed an anomalous alignment between the quadrupole ( $l = 2$ ) and octopole ( $l = 3$ ) modes [57]. Could such alignment be explained in terms of cosmological quantum relaxation?

The orientation of a multipole  $l$  may be defined by the method of maximisation of the ‘angular momentum dispersion’ [67]. Taking  $a_{lm}(\hat{\mathbf{n}})$  to be the harmonic coefficients  $a_{lm}$  expressed in a rotated coordinate system with  $z$ -axis

<sup>5</sup>The theoretical value of  $c_3$  depends on the (unknown) number  $M$  of excited states in the pre-inflationary era, but the actual value of  $c_3$  may be deduced simply by fitting (30) to the data.

in the direction  $\hat{\mathbf{n}}$ , one may consider

$$\sum_m m^2 |a_{lm}(\hat{\mathbf{n}})|^2 \quad (115)$$

for arbitrary  $\hat{\mathbf{n}}$  and find the axis  $\hat{\mathbf{n}} = \hat{\mathbf{n}}_l$  for which (115) is maximised. One may then take  $\hat{\mathbf{n}}_l$  to define the multipole ‘orientation’ (up to a sign).

If the CMB is statistically isotropic, the orientations  $\hat{\mathbf{n}}_l$  and  $\hat{\mathbf{n}}_{l'}$  of different multipoles  $l$  and  $l'$  will be statistically independent. Furthermore, each  $\hat{\mathbf{n}}_l$  will be a random variable drawn (independently) from a probability distribution such that all possible directions are equally likely. The random variable  $\hat{\mathbf{n}}_l \cdot \hat{\mathbf{n}}_{l'}$  will be uniformly distributed on  $[-1, 1]$ . If we do not distinguish between  $\hat{\mathbf{n}}$  and  $-\hat{\mathbf{n}}$  we may consider  $|\hat{\mathbf{n}}_l \cdot \hat{\mathbf{n}}_{l'}|$ , whose probability distribution will be uniform on  $[0, 1]$ . The data indicate that  $\hat{\mathbf{n}}_2$  and  $\hat{\mathbf{n}}_3$  are improbably aligned, with a value  $|\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_3| \simeq 0.9849$  [67]. For a statistically isotropic sky, such close alignment occurs with a probability of  $1 - 0.9849 \simeq 1/66$ . (Consistent results for the alignment are obtained [57] using an alternative method based on the multipole vector decomposition [68].)

If instead the primordial perturbations are not statistically isotropic, then the orientations  $\hat{\mathbf{n}}_l$  and  $\hat{\mathbf{n}}_{l'}$  of different multipoles need not be statistically independent and all possible directions for each  $\hat{\mathbf{n}}_l$  need not be equally likely. According to the scenario discussed in this paper, there will be statistical anisotropy at small  $k$  with approximate isotropy at large  $k$ . We are then more likely to find anomalous statistics for  $\hat{\mathbf{n}}_l$  at lower values of  $l$  than we are to find them at higher values of  $l$ .

To make this quantitative, we would have to consider primordial probability distributions that are consistent with our anisotropic power spectrum (52) and calculate the resulting probability for multipole alignment as a function of  $l$ . In particular, we would need to evaluate the probability of finding significant alignment at the lowest  $l$  values with no significant alignment at higher  $l$ . This would provide a further constraint on our model from the data.

In quantum equilibrium, of course, we expect no alignment for any  $l$ : the orientations  $\hat{\mathbf{n}}_l$  and  $\hat{\mathbf{n}}_{l+1}$  will be independent random variables distributed uniformly on the unit sphere and  $|\hat{\mathbf{n}}_l \cdot \hat{\mathbf{n}}_{l+1}|$  will be distributed uniformly on  $[0, 1]$ . In quantum nonequilibrium, however, for low  $l$  our non-vanishing matrix elements  $\langle a_{(l+1)m'}^* a_{lm} \rangle \neq 0$  indicate that  $\hat{\mathbf{n}}_l$  will not be statistically independent of  $\hat{\mathbf{n}}_{l+1}$  and so  $|\hat{\mathbf{n}}_l \cdot \hat{\mathbf{n}}_{l+1}|$  may be distributed non-uniformly on  $[0, 1]$ . We have seen that the averaged correlation  $\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle}$  decays rather rapidly as  $\sim l^{-5/2}$  or  $\sim l^{-3}$ , so we might reasonably expect significant alignment of  $\hat{\mathbf{n}}_l$  and  $\hat{\mathbf{n}}_{l+1}$  only at the lowest values of  $l$ .

A proper assessment, however, requires us to construct appropriate primordial distributions that are consistent with the anisotropic spectrum (52) and to evaluate the corresponding (nonequilibrium) probability distributions for  $|\hat{\mathbf{n}}_l \cdot \hat{\mathbf{n}}_{l+1}|$ . We leave this task for future work.

## 6 Conclusion

In this paper we have shown that cosmological quantum relaxation predicts an anisotropic primordial power spectrum  $\mathcal{P}_{\mathcal{R}}(k, \hat{\mathbf{k}})$  equal to the usual isotropic quantum equilibrium spectrum  $\mathcal{P}_{\mathcal{R}}^{\text{QT}}(k)$  modulated by an anisotropic deficit function  $\xi(k, \hat{\mathbf{k}})$ . In the limit of weak anisotropy we have derived an explicit expression for  $\xi(k, \hat{\mathbf{k}})$  given by (52). We have explored some of the potentially testable consequences. The function  $\xi(k, \hat{\mathbf{k}})$  is determined (up to arbitrary constants) by the functions (51), which are in turn determined by derivatives of the inverse-tangent power deficit (30) which was derived previously from numerical simulations. The lowest-order off-diagonal corrections to the CMB covariance matrix are similarly determined and at low  $l$  the  $l - (l + 1)$  inter-multipole correlations show a characteristic scaling with  $l$  that is related to the inverse-tangent deficit. Our anisotropic spectrum also predicts a residual statistical anisotropy in the limit of large  $l$ , for which we have derived an approximate consistency relation between the scaling of the  $l - (l + 1)$  correlations and the scaling of the angular power spectrum  $C_l$ . Furthermore, the  $l - (l + 1)$  correlations at large and small scales are found to be related. We have sketched how our results might be applied to understanding the anomalous alignment of very low- $l$  multipoles, though this requires further study.

The key physical argument leading to these results was presented in Section 3.3. The quantum relaxation dynamics is independent of the direction  $\hat{\mathbf{k}}$  of the wave vector for the relaxing field mode, hence we expect to find the same inverse-tangent deficit function (30) for different directions  $\hat{\mathbf{k}}$  but in general with  $\hat{\mathbf{k}}$ -dependent fitting coefficients  $c_1(\hat{\mathbf{k}})$ ,  $c_2(\hat{\mathbf{k}})$  and  $c_3(\hat{\mathbf{k}})$  (assuming there are different initial conditions for different  $\hat{\mathbf{k}}$ 's) resulting in an effective  $\hat{\mathbf{k}}$ -dependent or anisotropic deficit function  $\xi(k, \hat{\mathbf{k}})$ . By expanding the coefficients  $c_1$ ,  $c_2$ ,  $c_3$  as functions of  $\hat{\mathbf{k}}$  and assuming weak anisotropy, to lowest order we are able to derive an explicit expression for  $\xi(k, \hat{\mathbf{k}})$  and thereby relate the off-diagonal covariance matrix to (derivatives of) the inverse-tangent deficit function. That is the essence of the argument made in this paper, from which the predictions follow.

We have not considered how realistic it might be to test our predictions. At low  $l$  we predict certain scalings  $\sim l^{-5/2}$ ,  $l^{-3}$  for the  $m$ -averaged covariance matrix elements  $\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle}$ , while at high  $l$  we predict an approximate consistency relation  $\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle} \sim C_l/l^2$ ,  $C_l/l$  between the scaling of  $\overline{\langle a_{(l+1)m'}^* a_{lm} \rangle}$  and the scaling of the angular power spectrum  $C_l$ . Are we likely to find statistically-significant evidence for or against such scalings in the CMB data? The evidence for  $l - (l + 1)$  correlations themselves seems fairly significant, but we require statistical tests searching for a particular scaling of the correlations with  $l$ . Perhaps appropriate tests could be devised. Any resulting significance may be slight when considered alone, but the overall significance could be raised when combined with other related features such as the power

deficit and the small-scale anisotropies. It does not seem out of the question that, by combining these features and considering the predicted relationships between them, the overall statistical significance for or against cosmological quantum relaxation could turn out to be considerable. However, we leave a proper analysis for future work.

In our scenario there is a relationship between the  $l - (l + 1)$  correlations at large and small scales because the same unknown coefficients  $\Delta_{10}^3$  and  $\Delta_{11}^3$  appear in the large-scale covariance matrix elements (75) and in the small-scale covariance matrix elements (110). In future work it would be of interest to examine if the predicted relationship could play a role in explaining the reported correlations between directional asymmetries at large and small scales [2].

It is instructive to compare our results with some of the other theoretical work on statistical anisotropy in the CMB.

A systematic and model-independent discussion of anisotropic primordial power spectra of the general form (13) was presented by Armendariz-Picon [50], who considered in particular coefficients  $g_{LM}(k)$  that are power laws in  $k$ . Armendariz-Picon also considered anisotropies with a cut-off at some maximum multipole and for this case showed how to construct an unbiased estimator that could be used to measure the values of certain integrals which are related to our integrals (66). Ackerman, Carroll and Wise [51] considered imprints of a primordial preferred direction on the CMB, assuming a parity  $\mathbf{k} \rightarrow -\mathbf{k}$  symmetry so that the leading term in (13) has  $L = 2$  (yielding off-diagonal terms  $\langle a_{(l\pm 2)m}^* a_{lm} \rangle \neq 0$  in the covariance matrix), and furthermore they presented a model in which the corresponding coefficients are scale-invariant (independent of  $k$ ) in the region of interest. Similarly, Pullen and Kamionkowski [52] discussed CMB statistics for an anisotropic primordial spectrum in which the general form (13) is restricted to terms with  $L$  even (so that in particular there is no dipolar term with  $L = 1$ ). Ma, Efstathiou and Challinor [53] discussed the effects of a quadrupole ( $L = 2$ ) modulation of the primordial spectrum, with a power-law dependence on  $k$ , again assuming a parity-invariance to rule out odd values of  $L$ . A notable recurring feature of the cited work is the assumption that the lowest-order  $L = 1$  term (and indeed all odd- $L$  terms) may be discarded on grounds of parity symmetry. As a result, these models do not yield  $l - (l + 1)$  correlations in the CMB (for which there is some evidence). In contrast, when considering the possibility of primordial quantum nonequilibrium, we should bear in mind that the resulting departures from the Born rule can in principle be quite arbitrary. In pilot-wave dynamics, the initial probability distribution is a contingent fact about the world to be determined empirically; it is in principle unconstrained by physical law. Therefore, there are no physical grounds for imposing parity symmetry on primordial quantum nonequilibrium. It is then quite conceivable that terms with  $L = 1$  will be present and could even be dominant (as we have assumed for definiteness and simplicity). Furthermore, because of the intrinsic contingency of nonequilibrium, we would not expect it to manifest simply as an effective preferred direction but as a more complicated and generalised asymmetry in  $\mathbf{k}$ -space.

More recently, predictive models have been proposed that break parity symmetry. For example, refs. [15] and [17] find dipolar terms generated by primordial domain walls and by a direction-dependent inflaton field respectively. We note that the various available models are based on quite different physical mechanisms and make different predictions. For example, ref. [16] predicts both a power deficit and a statistical anisotropy arising from a pre-inflationary Kasner geometry, where at small scales the predicted spectrum converges to the standard isotropic form. This is in contrast with the scenario of cosmological quantum relaxation discussed here, for which we expect a residual statistical anisotropy even at small scales. Further comparison among the proposed models and with data is, however, a matter for future work.

Finally, it is worth noting that definitive observational evidence for statistical anisotropy in the primordial spectrum would not necessarily require us to abandon the inflationary paradigm for the origin of cosmic structure. As we have discussed, in the pilot-wave formulation of quantum theory the probability distribution is in principle independent of the quantum state. In particular, quantum nonequilibrium changes the spectrum of vacuum fluctuations without changing the vacuum wave functional itself. It is then straightforward to introduce primordial anisotropies without altering the essential structure and formalism of inflationary cosmology. Quantum relaxation ensures that the standard predictions are still obtained to a first approximation. To a second approximation, the suppression of quantum relaxation at large scales offers a single physical mechanism that might explain both the observed power deficit and the observed statistical anisotropy. We have shown that the dynamics of quantum relaxation leads to quantitative predictions for both kinds of anomaly as well as predicting relations between them. Whether these predictions will be confirmed or ruled out by the data remains to be seen.

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